

# Global well-posedness and large time asymptotic behavior of strong solutions to the 2-D compressible magnetohydrodynamic equations with vacuum

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## Abstract

The authors study the Cauchy problem of the magnetohydrodynamic equations for viscous compressible barotropic flows in two or three spatial dimensions with vacuum as far field density. For two spatial dimensions, we establish the global existence and uniqueness of strong solutions (which may be of possibly large oscillations) provided the smooth initial data are of small total energy, and obtain some a priori decay with rates (in large time) for the pressure, the spatial gradient of both the velocity field and the magnetic field. Moreover, for three spatial dimensions case, some similar decay rates are also obtained.

Keywords: compressible magnetohydrodynamic equations; global-wellposedness; large-time behavior; Cauchy problem; vacuum.

## 1 Introduction

We consider the magnetohydrodynamic (MHD) equations

$$\begin{cases} \rho_t + \operatorname{div}(\rho u) = 0, \\ (\rho u)_t + \operatorname{div}(\rho u \otimes u) + \nabla P(\rho) = \mu \Delta u + (\mu + \lambda) \nabla(\operatorname{div} u) + (\nabla \times H) \times H, \\ H_t - \nabla \times (u \times H) = -\nabla \times (\nu \nabla \times H), \quad \operatorname{div} H = 0, \end{cases} \quad (1.1)$$

for viscous compressible magnetohydrodynamics flows. Here,  $t \geq 0$  is time,  $x \in \mathbb{R}^2$  is the spatial coordinate, and  $\rho = \rho(x, t)$ ,  $u = (u^1, u^2)(x, t)$ ,  $H = (H^1, H^2)(x, t)$ , and

$$P(\rho) = R\rho^\gamma \quad (R > 0, \gamma > 1) \quad (1.2)$$

are the fluid density, velocity, magnetic field and pressure, respectively. Without loss of generality, we assumed that  $R = 1$ . The constant viscosity coefficients  $\mu$  and  $\lambda$  satisfy the physical restrictions:

$$\mu > 0, \quad \mu + \lambda \geq 0. \quad (1.3)$$

The constant  $\nu > 0$  is the resistivity coefficient which is inversely proportional to the electrical conductivity constant and acts as the magnetic diffusivity of magnetic fields.

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We consider the Cauchy problem for (1.1) with  $(\rho, u, H)$  vanishing at infinity (in some weak sense) with given initial data  $\rho_0$ ,  $u_0$ , and  $H_0$  as

$$\rho(x, 0) = \rho_0(x), \quad \rho u(x, 0) = \rho_0 u_0(x), \quad H(x, 0) = H_0, \quad x \in \mathbb{R}^2. \quad (1.4)$$

There have been huge literatures on the compressible MHD problem (1.1) by many physicists and mathematicians due to its physical importance, complexity, rich phenomena and mathematical challenges, see for example, [3, 4, 6–10, 15, 16, 22, 23, 30, 31, 34–36, 39, 40] and the references therein. In particular, if there is no electromagnetic effect, i.e.,  $H = 0$ , then (1.1) reduces to the compressible Navier-Stokes equations for barotropic flows, which have also been discussed in numerous studies, see for example, [5, 11–14, 17–21, 24–29, 32, 37, 38] and the references therein. The issues of well-posedness and dynamical behaviors of MHD system are rather complicated to investigate because of the strong coupling and interplay interaction between the fluid motion and the magnetic field. Now, we briefly recall some results concerned with the multi-dimensional compressible MHD equations which are more relatively with our problem. The local strong solutions to the compressible MHD with large initial data were obtained, by Vol’pert-Khudiaev [35] as the initial density is strictly positive and by Fan-Yu [10] as the initial density may contain vacuum, respectively. And recently, the local existence of strong and classical solutions to the two-dimensional compressible MHD equations with vacuum as far field density has been studied in [31]. The global existence of solutions to the compressible MHD equations were obtained in many works: Kawashima [23] firstly obtained the global existence when the initial data are close to a non-vacuum equilibrium in  $H^3$ -norm; Hu-Wang [15, 16] and Fan-Yu [9] proved the global existence of renormalized solutions under the general large initial data assumptions; For the case that the initial density is allowed to vanish and even has compact support, Li-Xu-Zhang [27] established the global existence and uniqueness of classical solutions with constant state as far field which could be either vacuum or nonvacuum to (1.1)-(1.4) in three-dimensional space with smooth initial data which are of small total energy but possibly large oscillations, which generalized the results of Huang-Li-Xin [19] for barotropic compressible Navier-Stokes equations to the compressible MHD ones. Moreover, it was also showed in [27] that for any  $p > 2$ , the following large-time behavior of the solution holds:

$$\lim_{t \rightarrow \infty} (\|P(\rho) - P(\tilde{\rho})\|_{L^p(\mathbb{R}^3)} + \|\nabla u\|_{L^2(\mathbb{R}^3)} + \|\nabla H\|_{L^2(\mathbb{R}^3)}) = 0 \quad (1.5)$$

where  $\tilde{\rho}$  is the constant far field density.

For two-dimensional problems, only in the case that the far field density is away from vacuum, the techniques of [27] can be modified directly since at this case, for any  $p \in [2, \infty)$ , the  $L^p$ -norm of a function  $u$  can be bounded by  $\|\rho^{1/2}u\|_{L^2}$  and  $\|\nabla u\|_{L^2}$ , and the similar results can be obtained. However, when the far field density is vacuum, it seems difficult to bound the  $L^p$ -norm of  $u$  by  $\|\rho^{1/2}u\|_{L^2}$  and  $\|\nabla u\|_{L^2}$  for any  $p \geq 1$ , so the global existence and large time behavior of strong or classical solutions to the Cauchy problem are much more subtle and remain open. Therefore, the main aim of this paper is to study the global existence and large time behavior of strong solutions to (1.1)-(1.4) in some homogeneous Sobolev spaces in two-dimensional space with vacuum as far field density. Although recently, for the two-dimensional Cauchy problem of barotropic compressible Navier-Stokes equations with vacuum as far field density, Li-Xin [24] obtained both the global existence of strong solutions and the decay rates of

the pressure and the gradient of velocity. However, their theory cannot be applied directly to the MHD ones.

Before stating the main results, we first explain the notations and conventions used throughout this paper. For  $R > 0$ , set

$$B_R \triangleq \{x \in \mathbb{R}^2 \mid |x| < R\}, \quad \int f dx \triangleq \int_{\mathbb{R}^2} f dx.$$

Moreover, for  $1 \leq r \leq \infty, k \geq 1$ , and  $\beta > 0$ , the standard homogeneous and inhomogeneous Sobolev spaces are defined as follows:

$$\begin{cases} L^r = L^r(\mathbb{R}^2), & D^{k,r} = D^{k,r}(\mathbb{R}^2) = \{v \in L^1_{\text{loc}}(\mathbb{R}^2) \mid \nabla^k v \in L^r(\mathbb{R}^2)\}, \\ D^1 = D^{1,2}, & W^{k,r} = W^{k,r}(\mathbb{R}^2), \quad H^k = W^{k,2}, \\ \dot{H}^\beta = \left\{ f : \mathbb{R}^2 \rightarrow \mathbb{R} \mid \|f\|_{\dot{H}^\beta}^2 = \int |\xi|^{2\beta} |\hat{f}(\xi)|^2 d\xi < \infty \right\}, \end{cases}$$

where  $\hat{f}$  is the Fourier transform of  $f$ . Next, we give the definition of strong solution to (1.1) as follows:

**Definition 1.1** *If all derivatives involved in (1.1) for  $(\rho, u, H)$  are regular distributions, and equations (1.1) hold almost everywhere in  $\mathbb{R}^2 \times (0, T)$ , then  $(\rho, u, H)$  is called a strong solution to (1.1).*

The initial total energy is defined as:

$$C_0 = \int_{\mathbb{R}^2} \left( \frac{1}{2} \rho_0 |u_0|^2 + \frac{1}{2} |H_0|^2 + \frac{1}{\gamma - 1} P(\rho_0) \right) dx.$$

Without loss of generality, assume that the initial density  $\rho_0$  satisfies

$$\int_{\mathbb{R}^2} \rho_0 dx = 1, \tag{1.6}$$

which implies that there exists a positive constant  $N_0$  such that

$$\int_{B_{N_0}} \rho_0 dx \geq \frac{1}{2} \int \rho_0 dx = \frac{1}{2}. \tag{1.7}$$

We can now state our main result in this paper, concerning the global existence of strong solutions.

**Theorem 1.1** *In addition to (1.6) and (1.7), suppose that the initial data  $(\rho_0, u_0, H_0)$  satisfy for any given numbers  $M > 0$ ,  $\bar{\rho} \geq 1$ ,  $a > 1$ ,  $q > 2$ , and  $\beta \in (1/2, 1]$ ,*

$$\begin{cases} 0 \leq \rho_0 \leq \bar{\rho}, & \bar{x}^a \rho_0 \in L^1 \cap H^1 \cap W^{1,q}, \\ (u_0, H_0) \in \dot{H}^\beta \cap D^1, & \rho_0^{1/2} u_0 \in L^2, \quad \bar{x}^{a/2} H_0 \in L^2, \end{cases} \tag{1.8}$$

and that

$$\|u_0\|_{\dot{H}^\beta} + \|H_0\|_{\dot{H}^\beta} + \|\rho_0 \bar{x}^a\|_{L^1} + \| |H_0|^2 \bar{x}^a \|_{L^1} \leq M, \tag{1.9}$$

where

$$\bar{x} \triangleq (e + |x|^2)^{1/2} \log^2(e + |x|^2). \tag{1.10}$$

Then there exists a positive constant  $\varepsilon$  depending on  $\mu, \lambda, \gamma, a, \nu, \bar{\rho}, \beta, N_0$ , and  $M$  such that if

$$C_0 \leq \varepsilon, \quad (1.11)$$

the problem (1.1)–(1.4) has a unique global strong solution  $(\rho, u, H)$  satisfying for any  $0 < T < \infty$ ,

$$0 \leq \rho(x, t) \leq 2\bar{\rho}, \quad (x, t) \in \mathbb{R}^2 \times [0, T], \quad (1.12)$$

$$\left\{ \begin{array}{l} \rho \in C([0, T]; L^1 \cap H^1 \cap W^{1,q}), \\ \bar{x}^a \rho \in L^\infty(0, T; L^1 \cap H^1 \cap W^{1,q}), \\ \sqrt{\rho}u, \nabla u, \bar{x}^{-1}u, \sqrt{t}\sqrt{\rho}u_t \in L^\infty(0, T; L^2), \\ H, H^2, H\bar{x}^{a/2}, \nabla H, \sqrt{t}H_t \in L^\infty(0, T; L^2), \\ \nabla u \in L^2(0, T; H^1) \cap L^{(q+1)/q}(0, T; W^{1,q}), \\ \nabla H \in L^2([0, T]; H^1), \\ \sqrt{t}\nabla u \in L^2(0, T; W^{1,q}), \\ \sqrt{\rho}u_t, \nabla H\bar{x}^{a/2}, \sqrt{t}\nabla u_t, \sqrt{t}\nabla H_t, \sqrt{t}\bar{x}^{-1}u_t \in L^2(\mathbb{R}^2 \times (0, T)), \end{array} \right. \quad (1.13)$$

and

$$\inf_{0 \leq t \leq T} \int_{B_{N_1(1+t) \log^\alpha(e+t)}} \rho(x, t) dx \geq \frac{1}{4}, \quad (1.14)$$

for any  $\alpha > 1$  and some positive constant  $N_1$  depending only on  $\alpha, N_0$ , and  $M$ . Moreover,  $(\rho, u, H)$  has the following decay rates, that is, for  $t \geq 1$ ,

$$\left\{ \begin{array}{l} \|\nabla H(\cdot, t)\|_{L^2} \leq Ct^{-\frac{1}{2}}, \\ \|\nabla u(\cdot, t)\|_{L^p} \leq C(p)t^{-1+1/p}, \quad \text{for } p \in [2, \infty), \\ \|P(\cdot, t)\|_{L^r} \leq C(r)t^{-1+1/r}, \quad \text{for } r \in (1, \infty), \\ \|\nabla \omega(\cdot, t)\|_{L^2} + \|\nabla F(\cdot, t)\|_{L^2} \leq Ct^{-1}, \end{array} \right. \quad (1.15)$$

where

$$\omega \triangleq \partial_1 u^2 - \partial_2 u^1, \quad F \triangleq (2\mu + \lambda)\operatorname{div} u - P - \frac{1}{2}|H|^2, \quad (1.16)$$

are respectively the vorticity and the so-called effective viscous flux, and  $C$  depends on  $\mu, \lambda, \nu, \gamma, a, \bar{\rho}, \beta, N_0$ , and  $M$ .

For the three-dimensional case, that is,  $\Omega = \mathbb{R}^3$ , we have the following results concerning the decay properties of the global classical solutions whose existence can be found in [27].

**Theorem 1.2** *Let  $\Omega = \mathbb{R}^3$ . For given numbers  $M > 0$ ,  $\bar{\rho} \geq 1$ ,  $\beta \in (1/2, 1]$ , and  $q \in (3, 6)$ , suppose that the initial data  $(\rho_0, u_0, H_0)$  satisfy*

$$\rho_0, P(\rho_0) \in H^2 \cap W^{2,q}, \quad P(\rho_0), \rho_0|u_0|^2 \in L^1, \quad u_0, H_0 \in \dot{H}^\beta, \quad \nabla u_0, \nabla H_0 \in H^1, \quad (1.17)$$

$$0 \leq \rho_0 \leq \bar{\rho}, \quad \|u_0\|_{\dot{H}^\beta} + \|H_0\|_{\dot{H}^\beta} \leq M, \quad (1.18)$$

and the compatibility condition

$$-\mu \Delta u_0 - (\mu + \lambda) \nabla \operatorname{div} u_0 - (\nabla \times H_0) \times H_0 + \nabla P(\rho_0) = \rho_0^{1/2} g, \quad (1.19)$$

for some  $g \in L^2$ . Moreover, if  $\gamma > 3/2$ , assume that

$$\rho_0 \in L^1. \quad (1.20)$$

Then there exists a positive constant  $\varepsilon$  depending on  $\mu, \lambda, \gamma, \nu, \bar{\rho}, \beta$ , and  $M$  such that if

$$C_0 \leq \varepsilon, \quad (1.21)$$

the Cauchy problem (1.1)-(1.4) has a unique global classical solution  $(\rho, u, H)$  in  $\mathbb{R}^3 \times (0, \infty)$  satisfying for any  $0 < \tau < T < \infty$ ,

$$0 \leq \rho(x, t) \leq 2\bar{\rho}, \quad x \in \mathbb{R}^3, t \geq 0, \quad (1.22)$$

$$\begin{cases} \rho \in C([0, T]; L^{3/2} \cap H^2 \cap W^{2,q}), \\ P \in C([0, T]; L^1 \cap H^2 \cap W^{2,q}), \quad u \in L^\infty(0, T; L^6), \\ \nabla u \in L^\infty(0, T; H^1) \cap L^2(0, T; H^2) \cap L^\infty(\tau, T; H^2 \cap W^{2,q}), \\ \nabla u_t \in L^2(0, T; L^2) \cap L^\infty(\tau, T; H^1) \cap H^1(\tau, T; L^2), \\ H \in C([0, T]; H^2) \cap L^\infty(\tau, T; H^3), \\ H_t \in C([0, T]; L^2) \cap H^1(\tau, T; L^2). \end{cases} \quad (1.23)$$

Moreover, for  $r \in (1, \infty)$ , there exist positive constants  $C(r)$  and  $C$  depending on  $\mu, \lambda, \gamma, \bar{\rho}, \beta$ , and  $M$  such that for  $t \geq 1$ ,

$$\begin{cases} \|\nabla H(\cdot, t)\|_{L^p} \leq Ct^{-1+(6-p)/(4p)}, \text{ for } p \in [2, 6], \\ \|\nabla u(\cdot, t)\|_{L^p} \leq Ct^{-1+1/p}, \text{ for } p \in [2, 6], \\ \|P(\cdot, t)\|_{L^r} \leq C(r)t^{-1+1/r}, \text{ for } r \in (1, \infty), \\ \|\nabla F(\cdot, t)\|_{L^2} + \|\nabla \omega(\cdot, t)\|_{L^2} \leq Ct^{-1}, \end{cases} \quad (1.24)$$

where  $F, \omega$  defined in (1.16), if  $\gamma > 3/2$ ,  $C(r)$  and  $C$  both depend on  $\|\rho_0\|_{L^1(\mathbb{R}^3)}$  also.

**Remark 1.1** When  $H = 0$ , i.e., there is no electromagnetic field effect, (1.1) turns to be the compressible Navier-Stokes equations, and Theorems 1.1 and 1.2 are the same as those results of Li-Xin [24]. Roughly speaking, we generalize the results of [24] to the compressible MHD equations.

**Remark 1.2** It should be noted here that the large time decay rate estimates (1.15) plays a crucial role in deriving the global existence of strong solutions to the two-dimensional problem (1.1)-(1.4), which is completely different from the three-dimensional case ([27]). More precisely, the global existence of classical solutions to (1.1)-(1.4) in [27] was achieved without any bounds on the decay rates of the solutions partially due to the a priori  $L^6$ -bounds on the velocity field and the magnetic field.

**Remark 1.3** We should point out that the large time asymptotic decay with rates of the global strong solutions, (1.15) and (1.24), yield in particular that the  $L^2$ -norm of the pressure, the gradient of the velocity and the magnetic decay in time with a rate  $t^{-1/2}$ , and the gradient of the vorticity and the effective viscous flux decay faster than themselves. As will be seen in the proof, the large time asymptotic decay are mainly controlled by the decay rate of the  $L^2$ -norm of the pressure. When  $H = 0$ , the large time decay rates (1.15) and (1.24) are the same as theirs in [24]. However, the decay rates of the magnetic field for large time in (1.15) and (1.24) are all new for both the two and three spatial dimensions compressible MHD equations.

**Remark 1.4** *Similar as [31], when the initial data  $(\rho_0, u_0, H_0)$  satisfy some additional regularity and compatibility conditions, following the same arguments as the proof of Theorem 1.1, we can also establish the global existence and uniqueness of the classical solutions to (1.1) and the same decay rates as (1.15).*

We now make some comments on the analysis of this paper. Note that for initial data in the class satisfying (1.8) and (1.9) except  $(u_0, H_0) \in \dot{H}^\beta$ , the local existence and uniqueness of strong and classical solutions to the Cauchy problem, (1.1)-(1.4), have been established recently in [31]. To extend the strong solution globally in time, one needs some global a priori estimates on smooth solutions to (1.1)-(1.4) in suitable higher norms. It turns out that the key issue here is to derive both the time-independent upper bound for the density and the time-depending higher norm estimates of the smooth solution  $(\rho, u, H)$ . To this end, on the one hand, we try to adapt some basic ideas used in [27]. However, new difficulties arise in the two-dimensional case, since the analysis in [27] relies heavily on the basic fact that, the  $L^6$ -norm of  $v \in D^1(\mathbb{R}^3)$  can be bounded by the  $L^2$ -norm of the gradient of  $v$  which fails for  $v \in D^1(\mathbb{R}^2)$ . On the other hand, compared with the two-dimensional compressible Navier-Stokes equations considered by Li-Xin ([24]), for the compressible MHD equations, the strong coupling between the velocity vector field and the magnetic field such as  $\nabla \times (u \times H)$  and  $(\nabla \times H) \times H$ , will bring out some new difficulties. Therefore, motivated by [24], using the  $L^1$ -integrability of the density, we try to obtain that the  $L^2$ -norm in both space and time of the pressure is time-independent (see (3.38)). However, the usually  $L^2$ -norm of  $H_t$  (or  $\Delta H$ ) cannot be directly estimated due to the strong coupled term between the velocity vector field and the magnetic field,  $\nabla \times (u \times H)$ . The key observation to overcome this difficulty is as follows: Instead of estimating the  $L^2$ -norm of  $H_t$  (or  $\Delta H$ ), we multiply the magnetic equations by  $\Delta H$  and  $H\Delta|H|^2$  respectively (see (3.19) and (3.37)), and succeed in controlling the coupled term  $\nabla \times (u \times H)$  by the gradient of both the velocity and the magnetic after integration by parts. This yields some new desired a priori estimates of the  $L^2$ -norm of  $|H||\Delta H|$  in both space and time. In fact, this is the first key observation of this paper. Next, our second key point is to get the  $H^1$ -norm of the effective viscous flux decays at the rate of  $t^{-1/2}$  for large time (see (3.125)). This is completed by driving the rates of decay for not only  $\nabla u$  and  $P$  (compared with [24, Lemma 3.4]) but also  $H$  and  $\nabla H$ . Indeed, we prove that the  $L^2$ -norm of  $|H|^2$  and  $|H||\nabla H|$  decay at the rates of  $t^{-1/2}$  and  $t^{-1}$  respectively (see (3.46)). Then, using the expansion rates of the essential support of the density (see (3.63) or [24, (3.39)]) for large time, we obtain the bound of the  $L^p$ -norm of the gradient of the effective viscous flux (see (3.120)). Based on these key ingredients, we are able to obtain the estimates on  $L^1(0, \min\{1, T\}; L^\infty(\mathbb{R}^2))$ -norm and the time-independent ones on  $L^4(\min\{1, T\}, T; L^\infty(\mathbb{R}^2))$ -norm of the effective viscous flux (see (3.126)). Then, motivated by [25], with the help of these estimates and Zlotnik's inequality (see Lemma 2.6), the desired time-uniform upper bound of the density is obtained, this is the key for global estimates of strong solutions. The next main step, following the similarly arguments as [17–19, 24, 27], is to bound the gradients of the density, velocity and magnetic. More precisely, such bounds can be obtained by solving a logarithm Gronwall's inequality based on a Beale-Kato-Majda type inequality (see Lemma 2.7) and the a priori estimates we have just derived, and moreover, such a derivation yields simultaneously also the bound for  $L^1(0, T; L^\infty(\mathbb{R}^2))$ -norm of the gradient of the velocity, see Lemma 4.1 and its proof. Finally, our third new observation of this paper is to obtain the  $L^2$ -norm of  $\bar{x}^{a/2}H$  and  $\bar{x}^{a/2}\nabla H$  (see (4.19) and (4.20)), which will be used in the estimates of the  $H^1$ -norm of the gradient

of both the velocity and the magnetic (see (4.29)).

The rest of the paper is organized as follows: In Section 2, we collect some elementary facts and inequalities which will be needed in later analysis. Sections 3 and 4 are devoted to deriving the necessary a priori estimates on strong solutions which are needed to extend the local solution to all time. Then finally, the main results, Theorems 1.1-1.2, are proved in Section 5.

## 2 Preliminaries

In this section, we will recall some known facts and elementary inequalities which will be used frequently later.

We begin with the local existence of strong and classical solutions whose proof can be found in [31].

**Lemma 2.1** *Assume that  $(\rho_0, u_0, H_0)$  satisfies (1.8) except  $(u_0, H_0) \in \dot{H}^\beta$ . Then there exist a small time  $T > 0$  and a unique strong solution  $(\rho, u, H)$  to the problem (1.1)-(1.4) in  $\mathbb{R}^2 \times (0, T)$  satisfying (1.13) and (1.14).*

Next, the following well-known Gagliardo-Nirenberg inequality (see [33]) will be used later.

**Lemma 2.2 (Gagliardo-Nirenberg)** *For  $p \in [2, \infty)$ ,  $q \in (1, \infty)$ , and  $r \in (2, \infty)$ , there exists some generic constant  $C > 0$  which may depend on  $p, q$ , and  $r$  such that for  $f \in H^1(\mathbb{R}^2)$  and  $g \in L^q(\mathbb{R}^2) \cap D^{1,r}(\mathbb{R}^2)$ , we have*

$$\|f\|_{L^p(\mathbb{R}^2)}^p \leq C \|f\|_{L^2(\mathbb{R}^2)}^2 \|\nabla f\|_{L^2(\mathbb{R}^2)}^{p-2}, \quad (2.1)$$

$$\|g\|_{C(\overline{\mathbb{R}^2})} \leq C \|g\|_{L^q(\mathbb{R}^2)}^{q(r-2)/(2r+q(r-2))} \|\nabla g\|_{L^r(\mathbb{R}^2)}^{2r/(2r+q(r-2))}. \quad (2.2)$$

The following weighted  $L^p$  bounds for elements of the Hilbert space  $D^1(\mathbb{R}^2)$  can be found in [28, Theorem B.1].

**Lemma 2.3** *For  $m \in [2, \infty)$  and  $\theta \in (1 + m/2, \infty)$ , there exists a positive constant  $C$  such that we have for all  $v \in D^{1,2}(\mathbb{R}^2)$ ,*

$$\left( \int_{\mathbb{R}^2} \frac{|v|^m}{e + |x|^2} (\log(e + |x|^2))^{-\theta} dx \right)^{1/m} \leq C \|v\|_{L^2(B_1)} + C \|\nabla v\|_{L^2(\mathbb{R}^2)}. \quad (2.3)$$

The following lemma was deduced in [24], we only state it here without proof.

**Lemma 2.4** *For  $\bar{x}$  as in (1.10), suppose that  $\rho \in L^\infty(\mathbb{R}^2)$  is a function such that*

$$0 \leq \rho \leq M_1, \quad M_2 \leq \int_{B_{N_*}} \rho dx, \quad \rho \bar{x}^\alpha \in L^1(\mathbb{R}^2), \quad (2.4)$$

for  $N_* \geq 1$  and positive constants  $M_1, M_2$ , and  $\alpha$ . Then, for  $r \in [2, \infty)$ , there exists a positive constant  $C$  depending only on  $M_1, M_2, \alpha$ , and  $r$  such that

$$\left( \int_{\mathbb{R}^2} \rho |v|^r dx \right)^{1/r} \leq C N_*^3 (1 + \|\rho \bar{x}^\alpha\|_{L^1(\mathbb{R}^2)}) \left( \|\rho^{1/2} v\|_{L^2(\mathbb{R}^2)} + \|\nabla v\|_{L^2(\mathbb{R}^2)} \right), \quad (2.5)$$

for each  $v \in \{v \in D^1(\mathbb{R}^2) \mid \rho^{1/2} v \in L^2(\mathbb{R}^2)\}$ .

Next, symbols  $\nabla^\perp \triangleq (-\partial_2, \partial_1)$ ,  $\dot{f} \triangleq f_t + u \cdot \nabla f$ , denoting the material derivative of  $f$ . We state some elementary estimates which follow from (2.1) and the standard  $L^p$ -estimate for the following elliptic system derived from the momentum equations in (1.1):

$$\Delta F = \operatorname{div}(\rho \dot{u} - \operatorname{div}(H \otimes H)), \quad \mu \Delta \omega = \nabla^\perp \cdot (\rho \dot{u} - \operatorname{div}(H \otimes H)), \quad (2.6)$$

where  $F$  and  $\omega$  are as in (1.16).

**Lemma 2.5** *Let  $(\rho, u, H)$  be a smooth solution of (1.1). Then for  $p \geq 2$  there exists a positive constant  $C$  depending only on  $p, \mu$ , and  $\lambda$  such that*

$$\|\nabla F\|_{L^p(\mathbb{R}^2)} + \|\nabla \omega\|_{L^p(\mathbb{R}^2)} \leq C(\|\rho \dot{u}\|_{L^p(\mathbb{R}^2)} + \|H\| \|\nabla H\|_{L^p(\mathbb{R}^2)}), \quad (2.7)$$

$$\begin{aligned} \|F\|_{L^p(\mathbb{R}^2)} + \|\omega\|_{L^p(\mathbb{R}^2)} &\leq C(\|\rho \dot{u}\|_{L^2(\mathbb{R}^2)} + \|H\| \|\nabla H\|_{L^2(\mathbb{R}^2)})^{1-2/p} \\ &\quad \cdot (\|\nabla u\|_{L^2(\mathbb{R}^2)} + \|P\|_{L^2(\mathbb{R}^2)} + \|H\|_{L^4}^2)^{2/p}, \end{aligned} \quad (2.8)$$

$$\begin{aligned} \|\nabla u\|_{L^p(\mathbb{R}^2)} &\leq C(\|\rho \dot{u}\|_{L^2(\mathbb{R}^2)} + \|H\| \|\nabla H\|_{L^2(\mathbb{R}^2)})^{1-2/p} \\ &\quad \cdot (\|\nabla u\|_{L^2(\mathbb{R}^2)} + \|P\|_{L^2(\mathbb{R}^2)} + \|H\|_{L^4}^2)^{2/p} + C\|P\|_{L^p(\mathbb{R}^2)} + C\|H\|^2_{L^p}. \end{aligned} \quad (2.9)$$

*Proof.* On one hand, the standard  $L^p$ -estimate for the elliptic system (2.6) yields (2.7) directly, which, together with (2.1) and (1.16), gives (2.8). On the other hand, since  $-\Delta u = -\nabla \operatorname{div} u - \nabla^\perp \omega$ , we have

$$\nabla u = -\nabla(-\Delta)^{-1} \nabla \operatorname{div} u - \nabla(-\Delta)^{-1} \nabla^\perp \omega. \quad (2.10)$$

Thus applying the standard  $L^p$ -estimate to (2.10) shows

$$\begin{aligned} \|\nabla u\|_{L^p(\mathbb{R}^2)} &\leq C(p)(\|\operatorname{div} u\|_{L^p(\mathbb{R}^2)} + \|\omega\|_{L^p(\mathbb{R}^2)}) \\ &\leq C(p)\|F\|_{L^p(\mathbb{R}^2)} + C(p)\|\omega\|_{L^p(\mathbb{R}^2)} + C(p)\|P\|_{L^p(\mathbb{R}^2)} + C\|H\|^2_{L^p}, \end{aligned}$$

which, along with (2.8), gives (2.9). Then, the proof of Lemma 2.5 is completed.

Next, in order to get the uniform (in time) upper bound of the density  $\rho$ , we need the following Zlotnik inequality.

**Lemma 2.6** ([41]) *Let the function  $y$  satisfy*

$$y'(t) = g(y) + b'(t) \text{ on } [0, T], \quad y(0) = y^0,$$

*with  $g \in C(\mathbb{R})$  and  $y, b \in W^{1,1}(0, T)$ . If  $g(\infty) = -\infty$  and*

$$b(t_2) - b(t_1) \leq N_0 + N_1(t_2 - t_1) \quad (2.11)$$

*for all  $0 \leq t_1 < t_2 \leq T$  with some  $N_0 \geq 0$  and  $N_1 \geq 0$ , then*

$$y(t) \leq \max\{y^0, \bar{\zeta}\} + N_0 < \infty \text{ on } [0, T],$$

*where  $\bar{\zeta}$  is a constant such that*

$$g(\zeta) \leq -N_1 \quad \text{for } \zeta \geq \bar{\zeta}. \quad (2.12)$$

Finally, the following Beale-Kato-Majda type inequality, which was proved in [1] when  $\operatorname{div} u \equiv 0$ , will be used later to estimate  $\|\nabla u\|_{L^\infty}$  and  $\|\nabla \rho\|_{L^2 \cap L^q}$  ( $q > 2$ ).

**Lemma 2.7** *For  $2 < q < \infty$ , there is a constant  $C(q)$  such that the following estimate holds for all  $\nabla u \in L^2(\mathbb{R}^2) \cap D^{1,q}(\mathbb{R}^2)$ ,*

$$\|\nabla u\|_{L^\infty(\mathbb{R}^2)} \leq C(\|\operatorname{div} u\|_{L^\infty(\mathbb{R}^2)} + \|\omega\|_{L^\infty(\mathbb{R}^2)}) \log(e + \|\nabla^2 u\|_{L^q(\mathbb{R}^2)}) + C\|\nabla u\|_{L^2(\mathbb{R}^2)} + C.$$



### 3 A priori estimates(I): lower order estimates

In this section, we will establish some necessary a priori bounds for smooth solutions to the Cauchy problem (1.1)-(1.4) to extend the local strong solution guaranteed by Lemma 2.1. Thus, let  $T > 0$  be a fixed time and  $(\rho, u, H)$  be the smooth solution to (1.1)-(1.4) on  $\mathbb{R}^2 \times (0, T]$  with smooth initial data  $(\rho_0, u_0, H_0)$  satisfying (1.8) and (1.9).

Set  $\sigma(t) \triangleq \min\{1, t\}$ . Define

$$A_1(T) \triangleq \sup_{0 \leq t \leq T} (\sigma \|\nabla u\|_{L^2}^2 + \sigma \|\nabla H\|_{L^2}^2) + \int_0^T \sigma \left( \|\rho^{1/2} \dot{u}\|_{L^2}^2 + \|\Delta H\|_{L^2}^2 \right) dt, \quad (3.1)$$

$$\begin{aligned} A_2(T) \triangleq & \sup_{0 \leq t \leq T} \sigma^2 \left( \|\rho^{1/2} \dot{u}\|_{L^2}^2 + \|H\| \|\nabla H\|_{L^2}^2 \right) \\ & + \int_0^T \sigma^2 \left( \|\nabla \dot{u}\|_{L^2}^2 + \|\Delta |H|^2\|_{L^2}^2 + \|\Delta H\| \|H\|_{L^2}^2 \right) dt, \end{aligned} \quad (3.2)$$

$$\begin{aligned} A_3(T) \triangleq & \sup_{0 \leq t \leq T} \sigma^{(3-2\beta)/4} (\|\nabla u\|_{L^2}^2 + \|\nabla H\|_{L^2}^2) \\ & + \int_0^T \sigma^{(3-2\beta)/4} (\|\rho^{1/2} \dot{u}\|_{L^2}^2 + \|\nabla^2 H\|_{L^2}^2) dt. \end{aligned} \quad (3.3)$$

We have the following key a priori estimates on  $(\rho, u, H)$ .

**Proposition 3.1** *Under the conditions of Theorem 1.1, there exists some positive constant  $\varepsilon$  depending on  $\mu, \lambda, \nu, \gamma, a, \bar{\rho}, \beta, N_0$ , and  $M$  such that if  $(\rho, u, H)$  is a smooth solution of (1.1)-(1.4) on  $\mathbb{R}^2 \times (0, T]$  satisfying*

$$\sup_{\mathbb{R}^2 \times [0, T]} \rho \leq 2\bar{\rho}, \quad A_1(T) + A_2(T) \leq 2C_0^{1/2}, \quad A_3(\sigma(T)) \leq 2C_0^{\delta_0}, \quad (3.4)$$

where  $\delta_0 = (2\beta - 1)/(9\beta)$ , the following estimates hold

$$\sup_{\mathbb{R}^2 \times [0, T]} \rho \leq 7\bar{\rho}/4, \quad A_1(T) + A_2(T) + \int_0^T \sigma \|P\|_{L^2}^2 dt \leq C_0^{1/2}, \quad A_3(\sigma(T)) \leq C_0^{\delta_0}, \quad (3.5)$$

provided  $C_0 \leq \varepsilon$ .

The proof of Proposition 3.1 will be postponed to the end of this section.

In the following, we will use the convention that  $C$  denotes a generic positive constant depending on  $\mu, \lambda, \nu, \gamma, a, \bar{\rho}, \beta, N_0$ , and  $M$ , and use  $C(\alpha)$  to emphasize that  $C$  depends on  $\alpha$ .

We first give the following standard energy estimate for  $(\rho, u, H)$ .

**Lemma 3.2** *Let  $(\rho, u, H)$  be a smooth solution of (1.1)-(1.4) on  $\mathbb{R}^2 \times (0, T]$ . Then there exists a positive constant depending only on  $\mu, \lambda, \nu$ , and  $\gamma$  that*

$$\sup_{0 \leq t \leq T} \int \left( \frac{1}{2} \rho |u|^2 + \frac{1}{\gamma - 1} P + \frac{1}{2} |H|^2 \right) dx + \int_0^T \int (\mu |\nabla u|^2 + \nu |\nabla H|^2) dx dt \leq C_0, \quad (3.6)$$

$$\sup_{0 \leq t \leq T} \|H\|_{L^2}^2 + \int_0^T \|H\| \|\nabla H\|_{L^2}^2 dt \leq C C_0. \quad (3.7)$$

*Proof.* Multiplying (1.1)<sub>2</sub> and (1.1)<sub>3</sub> by  $u$  and  $H$ , respectively, then adding the resultant equations together, and integrating it by parts over  $\mathbb{R}^2$ , we deduce that

$$\frac{d}{dt} \int \left( \frac{1}{2} \rho |u|^2 + \frac{1}{\gamma-1} P + \frac{1}{2} |H|^2 \right) + \int (\mu |\nabla u|^2 + \nu |\nabla H|^2) \leq 0, \quad (3.8)$$

then integrating over  $(0, T)$ , we finish the proof of (3.6).

Next, multiplying (1.1)<sub>3</sub> by  $|H|^2 H$ , then integrating over  $\mathbb{R}^2$  by parts, it holds

$$\begin{aligned} & \frac{1}{4} \frac{d}{dt} \| |H|^2 \|_{L^2}^2 + \nu (\| \nabla |H|^2 \|_{L^2}^2 + \| |H| \nabla H \|_{L^2}^2) \\ & \leq C \| \nabla u \|_{L^2} \| |H|^2 \|_{L^4}^2 \leq \frac{\nu}{2} \| \nabla |H|^2 \|_{L^2}^2 + C \| |H|^2 \|_{L^2}^2 \| \nabla u \|_{L^2}^2, \end{aligned} \quad (3.9)$$

thus the Gronwall's inequality and (3.6) shows

$$\sup_{0 \leq t \leq T} \| |H|^2 \|_{L^2}^2 + \int_0^T \| |H| \nabla H \|_{L^2}^2 dt \leq C \| H_0 \|_{L^4}^4 C_0 \leq C \| H_0 \|_{\dot{H}^\beta}^4 C_0$$

where we have used the Sobolev embedding inequality  $\dot{H}^\beta \hookrightarrow L^4$ . The proof of Lemma 3.2 is completed.

Next, We will prove the preliminary  $L^2$  bounds for  $\nabla u$ ,  $\nabla H$  and  $\rho \dot{u}$ .

**Lemma 3.3** *Let  $(\rho, u, H)$  be a smooth solution of (1.1)-(1.4) on  $\mathbb{R}^2 \times (0, T]$ . Then there exists a positive constant depending only on  $\mu$ ,  $\lambda$ ,  $\nu$ , and  $\gamma$  that*

$$A_1(T) \leq C C_0 + C \sup_{0 \leq t \leq T} \| P \|_{L^2}^2 + C \int_0^T \sigma \int (|\nabla u|^3 + |\nabla H|^3 + P |\nabla u|^2) dx dt, \quad (3.10)$$

$$A_2(T) \leq C C_0 + C A_1(T) + C \int_0^T \sigma^2 (\| \nabla u \|_{L^4}^4 + \| \nabla H \|_{L^4}^4 + \| P \|_{L^4}^4) dt. \quad (3.11)$$

*Proof. (Part I.)* First, we will prove (3.10). Multiplying (1.1)<sub>2</sub> by  $\dot{u}$  and then integrating the resulting equality over  $\mathbb{R}^2$  lead to

$$\begin{aligned} \int \rho |\dot{u}|^2 dx &= - \int \dot{u} \cdot \nabla P dx + \mu \int \Delta u \cdot \dot{u} dx + (\mu + \lambda) \int \nabla \operatorname{div} u \cdot \dot{u} dx \\ &\quad - \frac{1}{2} \int \dot{u} \cdot \nabla |H|^2 dx + \int H \cdot \nabla H \cdot \dot{u} dx. \end{aligned} \quad (3.12)$$

Since  $P$  satisfies

$$P_t + u \cdot \nabla P + \gamma P \operatorname{div} u = 0, \quad (3.13)$$

integration by parts yields that

$$\begin{aligned} - \int \dot{u} \cdot \nabla P dx &= \int ((\operatorname{div} u)_t P - (u \cdot \nabla u) \cdot \nabla P) dx \\ &= \left( \int \operatorname{div} u P dx \right)_t + \int ((\gamma - 1) P (\operatorname{div} u)^2 + P \partial_i u_j \partial_j u_i) dx \\ &\leq \left( \int \operatorname{div} u P dx \right)_t + C \int P |\nabla u|^2 dx. \end{aligned} \quad (3.14)$$

Integration by parts also implies that

$$\mu \int \Delta u \cdot \dot{u} dx \leq -\frac{\mu}{2} (\|\nabla u\|_{L^2}^2)_t + C \int |\nabla u|^3 dx, \quad (3.15)$$

and similarly

$$(\mu + \lambda) \int \nabla \operatorname{div} u \cdot \dot{u} dx \leq -\frac{\lambda + \mu}{2} (\|\operatorname{div} u\|_{L^2}^2)_t + C \int |\nabla u|^3 dx. \quad (3.16)$$

Following the same arguments in [31], we have

$$\begin{aligned} & \int H \cdot \nabla H \cdot \dot{u} dx - \frac{1}{2} \int \dot{u} \cdot \nabla |H|^2 dx \\ & \leq \frac{d}{dt} \left( \frac{1}{2} \int \operatorname{div} u |H|^2 dx - \int H \cdot \nabla u \cdot H dx \right) + C \|\nabla u\|_{L^3}^3 + C \|H\|_{L^6}^6 + \varepsilon \|\Delta H\|_{L^2}^2. \end{aligned} \quad (3.17)$$

Putting (3.14)-(3.17) into (3.12), it holds that

$$\begin{aligned} & \frac{d}{dt} \left( \frac{\mu}{2} \|\nabla u\|_{L^2}^2 + \frac{\lambda + \mu}{2} \|\operatorname{div} u\|_{L^2}^2 - \int \operatorname{div} u P dx - \frac{1}{2} \int \operatorname{div} u |H|^2 dx + \int H \cdot \nabla u \cdot H dx \right) \\ & + \int \rho |\dot{u}|^2 dx \leq C \int P |\nabla u|^2 dx + C \|\nabla u\|_{L^3}^3 + C \|H\|_{L^6}^6 + \varepsilon \|\Delta H\|_{L^2}^2. \end{aligned} \quad (3.18)$$

Next, multiplying (1.1)<sub>3</sub> by  $\Delta H$ , and integrating by parts over  $\mathbb{R}^2$ , we have

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int |\nabla H|^2 dx + \nu \int |\Delta H|^2 dx \leq C \int |\nabla u| |\nabla H|^2 dx + C \int |\nabla u| |H| |\Delta H| dx \\ & \leq C \|\nabla u\|_{L^3}^3 + C \|\nabla H\|_{L^3}^3 + C \|H\|_{L^6}^6 + \varepsilon \|\Delta H\|_{L^2}^2. \end{aligned} \quad (3.19)$$

Then adding (3.18) and (3.19) together, and choosing  $\varepsilon$  suitably small leads to

$$\begin{aligned} & B'(t) + \int (\rho |\dot{u}|^2 + |\Delta H|^2) dx \\ & \leq C \int P |\nabla u|^2 dx + C \|\nabla u\|_{L^3}^3 + C \|\nabla H\|_{L^3}^3 + C \|H\|_{L^6}^6 \\ & \leq C \int P |\nabla u|^2 dx + C \|\nabla u\|_{L^3}^3 + C \|\nabla H\|_{L^3}^3 + C \|\nabla H\|_{L^2}^2 + C \|H\|_{L^6}^6, \end{aligned} \quad (3.20)$$

where in the last inequality we have used

$$\|H\|_{L^6}^6 \leq C \|H\|_{L^4}^4 + C \|H\|_{L^4}^2 \|\Delta H\|_{L^2}^2 \leq C \|\nabla H\|_{L^2}^2 + C \|H\|_{L^6}^6 \|\nabla H\|_{L^2}^2 \quad (3.21)$$

owing to (3.6), (3.7), Gagliardo-Nirenberg, Hölder and Cauchy's inequalities, and

$$\begin{aligned} B(t) & \triangleq \frac{\mu}{2} \|\nabla u\|_{L^2}^2 + \frac{\lambda + \mu}{2} \|\operatorname{div} u\|_{L^2}^2 - \int \operatorname{div} u P dx \\ & + \frac{\nu}{2} \|\nabla H\|_{L^2}^2 + \int H \cdot \nabla u \cdot H dx - \frac{1}{2} \int \operatorname{div} u |H|^2 dx \end{aligned} \quad (3.22)$$

satisfies

$$\begin{aligned} & \frac{\mu}{4} \|\nabla u\|_{L^2}^2 + \frac{\nu}{2} \|\nabla H\|_{L^2}^2 - C \|P\|_{L^2}^2 - C \|H\|_{L^4}^4 \leq B(t) \\ & \leq C \|\nabla u\|_{L^2}^2 + C \|\nabla H\|_{L^2}^2 + C \|P\|_{L^2}^2 + C \|H\|_{L^4}^4. \end{aligned} \quad (3.23)$$

Then, integrating (3.20) multiplied by  $\sigma$  over  $(0, T)$  and using (3.23), (3.6) and (3.7) yield (3.10) directly.

**(Part II.)** We are now to prove (3.11). Operating  $\partial/\partial_t + \operatorname{div}(u \cdot \ )$  to  $(1.1)_2^j$  and multiplying the resulting equation by  $\dot{u}^j$ , one gets by some simple calculations that

$$\begin{aligned} \frac{1}{2} \left( \int \rho |\dot{u}^j|^2 dx \right)_t &= \int \rho_t |\dot{u}^j|^2 + \frac{1}{2} \rho (\dot{u}^j)_t^2 + \dot{u}^j \operatorname{div}(\rho \dot{u}^j u) dx \\ &= \mu \int \dot{u}^j (\Delta u_t^j + \operatorname{div}(u \Delta u^j)) dx + (\mu + \lambda) \int \dot{u}^j (\partial_t \partial_j (\operatorname{div} u) + \operatorname{div}(u \partial_j (\operatorname{div} u))) dx \\ &\quad - \int \dot{u}^j (\partial_j P_t + \operatorname{div}(u \partial_j P)) dx - \frac{1}{2} \int \dot{u}^j (\partial_t \partial_j |H|^2 + \operatorname{div}(u \partial_j |H|^2)) dx \\ &\quad + \int \dot{u}^j (\partial_t (H \cdot \nabla H^j) + \operatorname{div}(u (H \cdot \nabla H^j))) dx = \sum_{i=1}^5 R_i. \end{aligned} \quad (3.24)$$

Following the same argument as Hoff [13], we have the estimates of  $R_i$  ( $i = 1, 2, 3$ ) as follows

$$\begin{aligned} R_1 + R_2 + R_3 &\leq -\mu \int |\nabla \dot{u}|^2 dx - (\mu + \lambda) \int |\operatorname{div} \dot{u}|^2 dx \\ &\quad + \varepsilon \int |\nabla \dot{u}|^2 dx + C \int |\nabla u|^4 dx + C \int |P|^4 dx. \end{aligned} \quad (3.25)$$

Now, we estimate the term  $R_4$ . By (3.7), (1.1)<sub>3</sub> and (2.1), we have

$$\begin{aligned} R_4 &= \int \partial_j \dot{u}^j H \cdot H_t dx + \frac{1}{2} \int \partial_i \dot{u}^j u^i \partial_j H^2 dx \\ &= \frac{1}{2} \int \partial_j \dot{u}^j \partial_i u^i H^2 dx - \frac{1}{2} \int \partial_i \dot{u}^j \partial_j u^i H^2 dx \\ &\quad + \int \partial_j \dot{u}^j H \cdot (H \cdot \nabla u + \nu \Delta H - H \operatorname{div} u) dx \\ &\leq C \int |\nabla \dot{u}| |\nabla u| H^2 dx + C \int |\nabla \dot{u}| |\Delta H \cdot H| dx \\ &\leq \varepsilon \int |\nabla \dot{u}|^2 dx + C \int |\nabla u|^4 dx + C \int |H|^8 dx + C \int |\Delta H|^2 |H|^2 dx \end{aligned} \quad (3.26)$$

Similar to (3.26), we also have

$$R_5 \leq \varepsilon \int |\nabla \dot{u}|^2 dx + C \int |\nabla u|^4 dx + C \int |H|^8 dx + C \int |\Delta H|^2 |H|^2 dx. \quad (3.27)$$

Putting (3.25)-(3.27) into (3.24), and choosing  $\varepsilon$  suitably small yields that

$$\begin{aligned} \frac{1}{2} \left( \int \rho |\dot{u}|^2 dx \right)_t + \mu \int |\nabla \dot{u}|^2 dx \\ \leq C \|\nabla u\|_{L^4}^4 + C \|P\|_{L^4}^4 + C \| |H|^2 \|_{L^4}^4 + C \| |\Delta H| |H| \|_{L^2}^2. \end{aligned} \quad (3.28)$$

Next, following the similar arguments as [31], we will to get the estimate of the last term of (3.28). For  $a_1, a_2 \in \{-1, 0, 1\}$ , denote

$$\tilde{H}(a_1, a_2) = a_1 H^1 + a_2 H^2, \quad \tilde{u}(a_1, a_2) = a_1 u^1 + a_2 u^2. \quad (3.29)$$

It thus follows from (1.1)<sub>3</sub> that

$$\tilde{H}_t - \nu \Delta \tilde{H} = H \cdot \nabla \tilde{u} - u \cdot \nabla \tilde{H} + \tilde{H} \operatorname{div} u \quad (3.30)$$

which multiplied by  $\tilde{H}\Delta|\tilde{H}|^2$ , then integrating the resultant equation over  $\mathbb{R}^2$  leads to

$$\begin{aligned} \frac{1}{4} \left( \|\nabla|\tilde{H}|^2\|_{L^2}^2 \right)_t + \frac{\nu}{2} \|\Delta|\tilde{H}|^2\|_{L^2}^2 &= \nu \int |\nabla\tilde{H}|^2 \Delta|\tilde{H}|^2 dx - \int H \cdot \nabla \tilde{u} \tilde{H} \Delta|\tilde{H}|^2 dx \\ &+ \int \operatorname{div} u |\tilde{H}|^2 \Delta|\tilde{H}|^2 dx + \frac{1}{2} \int u \cdot \nabla |\tilde{H}|^2 \Delta|\tilde{H}|^2 dx \triangleq \check{J}_1 + \check{J}_2 + \check{J}_3 + \check{J}_4, \end{aligned} \quad (3.31)$$

where

$$\begin{aligned} \check{J}_1 + \check{J}_2 + \check{J}_3 &\leq C \|\nabla u\|_{L^4}^4 + C \|\nabla H\|_{L^4}^4 + C \| |H|^2 \|_{L^4}^4 + \varepsilon \|\Delta|\tilde{H}|^2\|_{L^2}^2, \\ \check{J}_4 &= -\frac{1}{2} \int \nabla u \cdot \nabla |\tilde{H}|^2 \cdot \nabla |\tilde{H}|^2 dx + \frac{1}{4} \int \operatorname{div} u |\nabla|\tilde{H}|^2|^2 dx \\ &\leq C \int |\nabla u| |\nabla\tilde{H}|^2 |\tilde{H}|^2 dx \leq C \|\nabla u\|_{L^4}^4 + C \|\nabla H\|_{L^4}^4 + C \| |H|^2 \|_{L^4}^4. \end{aligned} \quad (3.32)$$

Putting (3.32) into (3.31) and choosing  $\varepsilon$  suitably small yields that

$$\frac{d}{dt} \left( \|\nabla|\tilde{H}|^2\|_{L^2}^2 \right) + \|\Delta|\tilde{H}|^2\|_{L^2}^2 \leq C \|\nabla u\|_{L^4}^4 + C \|\nabla H\|_{L^4}^4 + C \| |H|^2 \|_{L^4}^4, \quad (3.33)$$

which multiplied by  $\sigma^2$ , and integrating the resultant inequality over  $(0, T)$ , we have by (3.7) that

$$\sup_{0 \leq t \leq T} \sigma^2 \left( \|\nabla|\tilde{H}|^2\|_{L^2}^2 \right) + \int_0^T \sigma^2 \|\Delta|\tilde{H}|^2\|_{L^2}^2 dt \leq CC_0 + C \int_0^T \sigma^2 (\|\nabla u\|_{L^4}^4 + \|\nabla H\|_{L^4}^4) dt. \quad (3.34)$$

Moreover, by some directly calculations, we have

$$\begin{aligned} \|\nabla H\|H\|_{L^2}^2 &\leq \check{C} \|\nabla|\tilde{H}|^2(1, 0)\|_{L^2}^2 + \check{C} \|\nabla|\tilde{H}|^2(0, 1)\|_{L^2}^2 \\ &+ \check{C} \|\nabla|\tilde{H}|^2(1, 1)\|_{L^2}^2 + \check{C} \|\nabla|\tilde{H}|^2(1, -1)\|_{L^2}^2, \end{aligned} \quad (3.35)$$

$$\begin{aligned} \|\Delta H\|H\|_{L^2}^2 &\leq C \|\nabla H\|_{L^4}^4 + \check{C} \|\Delta|\tilde{H}|^2(1, 0)\|_{L^2}^2 + \check{C} \|\Delta|\tilde{H}|^2(0, 1)\|_{L^2}^2 \\ &+ \check{C} \|\Delta|\tilde{H}|^2(1, 1)\|_{L^2}^2 + \check{C} \|\Delta|\tilde{H}|^2(1, -1)\|_{L^2}^2, \end{aligned} \quad (3.36)$$

(the constant  $\check{C}$  will be used in following (3.56)) which together with (3.34) loads

$$\begin{aligned} \sup_{0 \leq t \leq T} \left( \sigma^2 \|\nabla H\|H\|_{L^2}^2 \right) + \int_0^T \sigma^2 (\|\Delta|\tilde{H}|^2\|_{L^2}^2 + \|\Delta H\|H\|_{L^2}^2) dt \\ \leq CC_0 + C \int_0^T \sigma^2 (\|\nabla u\|_{L^4}^4 + \|\nabla H\|_{L^4}^4) dt. \end{aligned} \quad (3.37)$$

Finally, multiplying (3.28) by  $\sigma^2$ , we obtain by (3.37) that (3.11) is true. The proof of Lemma 3.3 is completed.

The next result shows that pressure decays in time.

**Lemma 3.4** *Let  $(\rho, u, H)$  be a smooth solution of (1.1)-(1.4) on  $\mathbb{R}^2 \times (0, T]$  satisfying (3.4). Then there exists a positive constant  $C(\bar{\rho})$  depending only on  $\mu, \lambda, \nu, \gamma$ , and  $\bar{\rho}$  such that*

$$A_1(T) + A_2(T) + \int_0^T \sigma \|P\|_{L^2}^2 dt \leq C(\bar{\rho})C_0. \quad (3.38)$$

*Proof.* First, it follows from (2.9), (3.4), (3.6), and (3.7) that

$$\begin{aligned}
& \int_0^T \sigma^2 (\|\nabla u\|_{L^4}^4 + \|\nabla H\|_{L^4}^4 + \|P\|_{L^4}^4) dt \\
& \leq C \int_0^T \sigma (\|\rho \dot{u}\|_{L^2}^2 + \|H\|_{L^2} \|\nabla H\|_{L^2}^2) (\sigma \|\nabla u\|_{L^2}^2 + \sigma \|P\|_{L^2}^2 + \sigma \|H\|_{L^4}^4) dt \\
& \quad + C \int_0^T \sigma^2 \|H\|_{L^4}^4 dt + C \int_0^T \sigma^2 \|\nabla H\|_{L^2}^2 \|\nabla^2 H\|_{L^2}^2 dt + C \int_0^T \sigma^2 \|P\|_{L^4}^4 dt \\
& \leq C(\bar{\rho}) (A_1(T) + C_0) \int_0^T (\sigma \|\rho^{1/2} \dot{u}\|_{L^2}^2 + \sigma \|H\|_{L^2} \|\nabla H\|_{L^2}^2) dt \\
& \quad + CC_0 \int_0^T \sigma \|\nabla |H|^2\|_{L^2}^2 dt + CA_1(T) \int_0^T \sigma \|\nabla^2 H\|_{L^2}^2 dt + C(\bar{\rho}) \int_0^T \sigma^2 \|P\|_{L^2}^2 dt \\
& \leq C(\bar{\rho}) (A_1(T) + C_0) \int_0^T (\sigma \|\rho^{1/2} \dot{u}\|_{L^2}^2 + \sigma \|\nabla^2 H\|_{L^2}^2 + \sigma \|H\|_{L^2} \|\nabla H\|_{L^2}^2) dt \\
& \quad + C(\bar{\rho}) \int_0^T \sigma^2 \|P\|_{L^2}^2 dt.
\end{aligned} \tag{3.39}$$

To estimate the last term on the right-hand side of (3.39), noticing that (1.1)<sub>2</sub> gives

$$P = (-\Delta)^{-1} \operatorname{div}(\rho \dot{u}) + (2\mu + \lambda) \operatorname{div} u + (-\Delta)^{-1} \operatorname{div} \operatorname{div}((H \otimes H) - \frac{1}{2} |H|^2), \tag{3.40}$$

we obtain from (3.6), Hölder's and Sobolev's inequalities that

$$\begin{aligned}
\int P^2 dx & \leq C \|(-\Delta)^{-1} \operatorname{div}(\rho \dot{u})\|_{L^{4\gamma}} \|P\|_{L^{4\gamma/(4\gamma-1)}} + C \|\nabla u\|_{L^2} \|P\|_{L^2} + C \|H\|_{L^2}^2 \|P\|_{L^2} \\
& \leq C \|\rho \dot{u}\|_{L^{4\gamma/(2\gamma+1)}} \|\rho\|_{L^1}^{1/2} \|\rho\|_{L^{2\gamma}}^{\gamma-1/2} + C \|\nabla u\|_{L^2} \|P\|_{L^2} + C \|H\|_{L^2} \|\nabla H\|_{L^2} \|P\|_{L^2} \\
& \leq C \|\rho^{1/2}\|_{L^{4\gamma}} \|\rho^{1/2} \dot{u}\|_{L^2} \|\rho\|_{L^1}^{1/2} \|\rho\|_{L^{2\gamma}}^{\gamma-1/2} + C \|\nabla u\|_{L^2} \|P\|_{L^2} + C \|\nabla H\|_{L^2} \|P\|_{L^2} \\
& \leq C \|P\|_{L^2} \|\rho^{1/2} \dot{u}\|_{L^2} + C \|\nabla u\|_{L^2} \|P\|_{L^2} + C \|\nabla H\|_{L^2} \|P\|_{L^2},
\end{aligned}$$

where in the last inequality, one has used

$$\int \rho dx = \int \rho_0 dx = 1, \tag{3.41}$$

due to the mass conservation equation (1.1)<sub>1</sub>. Thus, we arrive at

$$\|P\|_{L^2} \leq C \|\rho^{1/2} \dot{u}\|_{L^2} + C \|\nabla u\|_{L^2} + C \|\nabla H\|_{L^2}, \tag{3.42}$$

which, along with (3.10), (3.11), (3.39), (3.6), and (3.4) gives

$$A_1(T) + A_2(T) \leq C(\bar{\rho}) C_0 + C(\bar{\rho}) \int_0^T (\sigma \|\nabla u\|_{L^3}^3 + \sigma \|\nabla H\|_{L^3}^3) dt. \tag{3.43}$$

Then, on the one hand, one deduces from (2.9), (2.1), (3.6), (3.7), and (3.4) that

$$\begin{aligned}
& \int_0^{\sigma(T)} (\sigma \|\nabla u\|_{L^3}^3 + \sigma \|\nabla H\|_{L^3}^3) dt \\
& \leq C \int_0^{\sigma(T)} \sigma \left( \|\rho^{1/2} \dot{u}\|_{L^2} + \|H\|_{L^2} \|\nabla H\|_{L^2} \right) (\|\nabla u\|_{L^2}^2 + \|P\|_{L^2}^2 + \|H\|_{L^4}^4) dt \\
& \quad + C \int_0^{\sigma(T)} \sigma \|P\|_{L^3}^3 dt + C \int_0^{\sigma(T)} \sigma \|\nabla H\|_{L^2}^2 \|\nabla^2 H\|_{L^2} dt \\
& \leq C A_2^{1/2}(\sigma(T)) \int_0^{\sigma(T)} (\|\nabla u\|_{L^2}^2 + \|P\|_{L^2}^2 + \|\nabla H\|_{L^2}^2) dt \\
& \quad + C \int_0^{\sigma(T)} \|P\|_{L^3}^3 dt + C \int_0^{\sigma(T)} \sigma \|\nabla H\|_{L^2}^4 dt + \delta \int_0^{\sigma(T)} \sigma \|\nabla^2 H\|_{L^2}^2 dt \\
& \leq C(\bar{\rho}) C_0 + C A_1(\sigma(T)) \int_0^{\sigma(T)} \|\nabla H\|_{L^2}^2 dt + \delta A_1(T) \\
& \leq C(\bar{\rho}) C_0 + \delta A_1(T).
\end{aligned} \tag{3.44}$$

On the other hand, Hölder's inequality, (3.39), (3.4), and (3.42) imply

$$\begin{aligned}
& \int_{\sigma(T)}^T (\sigma \|\nabla u\|_{L^3}^3 + \sigma \|\nabla H\|_{L^3}^3) dt \\
& \leq \delta \int_{\sigma(T)}^T \sigma^2 (\|\nabla u\|_{L^4}^4 + \|\nabla H\|_{L^4}^4) dt + C(\delta) \int_{\sigma(T)}^T (\|\nabla u\|_{L^2}^2 + \|\nabla H\|_{L^2}^2) dt \\
& \leq \delta C(\bar{\rho}) A_1(T) + C(\delta) C(\bar{\rho}) C_0.
\end{aligned} \tag{3.45}$$

Finally, putting (3.44) and (3.45) into (3.43) and choosing  $\delta$  suitably small lead to

$$A_1(T) + A_2(T) \leq C(\bar{\rho}) C_0,$$

which together with (3.42) and (3.6) gives (3.38) and finishes the proof of Lemma 3.4.

Next, we derive the rates of decay for  $\nabla u, \nabla H, H$  and  $P$ , which are essential to obtain the uniform (in time) upper bound of the density for large time.

**Lemma 3.5** *For  $p \in [2, \infty)$ , there exists a positive constant  $C(p, \bar{\rho})$  depending only on  $p, \mu, \nu, \lambda, \gamma$ , and  $\bar{\rho}$  such that, if  $(\rho, u, H)$  is a smooth solution of (1.1)–(1.4) on  $\mathbb{R}^2 \times (0, T]$  satisfying (3.4), then*

$$\begin{aligned}
& \sup_{\sigma(T) \leq t \leq T} \left[ t^{p-1} (\|\nabla u\|_{L^p}^p + \|P\|_{L^p}^p) + t (\|\nabla H\|_{L^2}^2 + \|H\|^2_{L^2}) \right] \\
& + \sup_{\sigma(T) \leq t \leq T} \left[ t^2 \left( \|\rho^{1/2} \dot{u}\|_{L^2}^2 + \|\nabla H\|_{L^2} \|\nabla^2 H\|_{L^2} \right) \right] \leq C(\bar{\rho}) C_0.
\end{aligned} \tag{3.46}$$

*Proof.* First, for  $p \geq 2$ , multiplying (3.13) by  $pP^{p-1}$  and integrating the resulting equality over  $\mathbb{R}^2$ , one gets after using  $\operatorname{div} u = \frac{1}{2\mu+\lambda} (F + P + \frac{1}{2}|H|^2)$  that

$$\begin{aligned}
& (\|P\|_{L^p}^p)_t + \frac{p\gamma-1}{2\mu+\lambda} \|P\|_{L^{p+1}}^{p+1} = -\frac{p\gamma-1}{2\mu+\lambda} \int P^p \left( F + \frac{1}{2}|H|^2 \right) dx \\
& \leq \frac{p\gamma-1}{2(2\mu+\lambda)} \|P\|_{L^{p+1}}^{p+1} + C(p) \|F\|_{L^{p+1}}^{p+1} + C(p) \|H\|^2_{L^{p+1}}^{p+1},
\end{aligned} \tag{3.47}$$

which together with (2.8) and (2.1) gives

$$\begin{aligned} & \frac{2(2\mu + \lambda)}{p\gamma - 1} (\|P\|_{L^p}^p)_t + \|P\|_{L^{p+1}}^{p+1} \leq C(p) \|F\|_{L^{p+1}}^{p+1} + C(p) \| |H|^2 \|_{L^{p+1}}^{p+1} \\ & \leq C(p) (\|\nabla u\|_{L^2}^2 + \|P\|_{L^2}^2 + \|H\|_{L^4}^4) (\|\rho \dot{u}\|_{L^2} + \| |H| |\nabla H| \|_{L^2})^{p-1}. \end{aligned} \quad (3.48)$$

In particular, choosing  $p = 2$  in (3.48) shows

$$\begin{aligned} & (\|P\|_{L^2}^2)_t + \frac{2\gamma - 1}{2(2\mu + \lambda)} \|P\|_{L^3}^3 \\ & \leq \delta \|\rho^{1/2} \dot{u}\|_{L^2}^2 + \delta \| |H| |\nabla H| \|_{L^2}^2 + C(\delta) (\|\nabla u\|_{L^2}^4 + \|\nabla H\|_{L^2}^4 + \|P\|_{L^2}^4), \end{aligned} \quad (3.49)$$

where in the last inequality we have used (2.1) and (3.6).

Next, it follows from (3.20), (2.9), (2.1) and (3.6) that

$$\begin{aligned} & B'(t) + \int \rho |\dot{u}|^2 dx + \int |\Delta H|^2 dx \\ & \leq C \|P\|_{L^3}^3 + C \|\nabla u\|_{L^3}^3 + C \|\nabla H\|_{L^3}^3 + C \| |H|^2 \|_{L^3}^3 \\ & \leq C_1 \|P\|_{L^3}^3 + C (\|\rho \dot{u}\|_{L^2} + \| |H| |\nabla H| \|_{L^2}) (\|\nabla u\|_{L^2}^2 + \|P\|_{L^2}^2 + \|H\|_{L^4}^4) \\ & \quad + C \|\nabla H\|_{L^2}^2 \|\nabla^2 H\|_{L^2}^2 \\ & \leq C_1 \|P\|_{L^3}^3 + \delta \|\rho^{1/2} \dot{u}\|_{L^2}^2 + \delta \| |H| |\nabla H| \|_{L^2}^2 + \delta \|\Delta H\|_{L^2}^2 \\ & \quad + C(\bar{\rho}, \delta) (\|\nabla u\|_{L^2}^4 + \|\nabla H\|_{L^2}^4 + \|P\|_{L^2}^4). \end{aligned} \quad (3.50)$$

Moreover, by (3.9), (2.1), (3.6) and Cauchy inequality, it holds that

$$\frac{d}{dt} \|H\|_{L^4}^4 + \| |H| |\nabla H| \|_{L^2}^2 \leq C \|\nabla u\|_{L^2}^4 + C \|\nabla H\|_{L^2}^4. \quad (3.51)$$

Choosing  $C_2 \geq 2 + 2(2\mu + \lambda)(C_1 + 1)/(2\gamma - 1)$  and  $C_3$  suitably large such that

$$\begin{aligned} & \frac{\mu}{4} \|\nabla u\|_{L^2}^2 + \frac{\nu}{2} \|\nabla H\|_{L^2}^2 + \|P\|_{L^2}^2 \leq B(t) + C_2 \|P\|_{L^2}^2 + C_3 \|H\|_{L^4}^4 \\ & \leq C \|\nabla u\|_{L^2}^2 + C \|\nabla H\|_{L^2}^2 + C \|P\|_{L^2}^2, \end{aligned} \quad (3.52)$$

then adding (3.49) multiplied by  $C_2$  and (3.51) multiplied by  $C_3$  to (3.50), after choosing  $\delta$  suitably small, it holds that

$$\begin{aligned} & 2 (B(t) + C_2 \|P\|_{L^2}^2 + C_3 \|H\|_{L^4}^4)' + \int (\rho |\dot{u}|^2 + |\Delta H|^2 + |H|^2 |\nabla H|^2 + P^3) dx \\ & \leq C \|P\|_{L^2}^4 + C \|\nabla u\|_{L^2}^4 + C \|\nabla H\|_{L^2}^4, \end{aligned} \quad (3.53)$$

which multiplied by  $t$ , together with Gronwall's inequality, (3.52), (3.38), (3.6), and (3.4) yields

$$\begin{aligned} & \sup_{\sigma(T) \leq t \leq T} t (\|\nabla u\|_{L^2}^2 + \|\nabla H\|_{L^2}^2 + \|P\|_{L^2}^2 + \|H\|_{L^4}^4) \\ & + \int_{\sigma(T)}^T t \int (\rho |\dot{u}|^2 + |\Delta H|^2 + |H|^2 |\nabla H|^2 + P^3) dx dt \leq C(\bar{\rho}) C_0. \end{aligned} \quad (3.54)$$

Next, multiplying (3.28) by  $t^2$ , we have

$$\begin{aligned} & \left( t^2 \int \rho |\dot{u}|^2 dx \right)_t + \mu t^2 \int |\nabla \dot{u}|^2 dx \leq 2t \int \rho |\dot{u}|^2 dx + Ct^2 \|\nabla u\|_{L^4}^4 + Ct^2 \|P\|_{L^4}^4 \\ & \quad + Ct^2 \| |H|^2 \|_{L^4}^4 + \hat{C} t^2 \| |\Delta H| |H| \|_{L^2}^2, \end{aligned} \quad (3.55)$$



and multiplying (3.34) by  $\check{C}(\hat{C} + 1)t^2$  ( $\check{C}$  is defined in (3.35) and (3.36)) yields that

$$\begin{aligned} & \frac{d}{dt} \left( \check{C}(\hat{C} + 1)t^2 \|\nabla|\tilde{H}|^2\|_{L^2}^2 \right) + \check{C}(\hat{C} + 1)t^2 \|\Delta|\tilde{H}|^2\|_{L^2}^2 \\ & \leq Ct^2 (\|\nabla u\|_{L^4}^4 + \|\nabla H\|_{L^4}^4 + \| |H|^2 \|_{L^4}^4) + Ct \|\nabla H\|_H \|\nabla H\|_{L^2}^2. \end{aligned} \quad (3.56)$$

Adding (3.56) to (3.55) implies that

$$\begin{aligned} & \left( t^2 \int \rho |\dot{u}|^2 dx + \check{C}(\hat{C} + 1)t^2 \|\nabla|\tilde{H}|^2\|_{L^2}^2 \right)_t \\ & + \mu t^2 \|\nabla \dot{u}\|_{L^2}^2 + \check{C}(\hat{C} + 1)t^2 \|\Delta|\tilde{H}|^2\|_{L^2}^2 \\ & \leq \hat{C}t^2 \|\Delta H\|_H \|\nabla H\|_{L^2}^2 + Ct \left( \|\nabla H\|_H \|\nabla H\|_{L^2}^2 + \|\rho^{1/2} \dot{u}\|_{L^2}^2 \right) \\ & + Ct^2 (\|\nabla u\|_{L^4}^4 + \|\nabla H\|_{L^4}^4 + \| |H|^2 \|_{L^4}^4) + Ct^2 \|P\|_{L^4}^4 \\ & \leq \hat{C}t^2 \|\Delta H\|_H \|\nabla H\|_{L^2}^2 + Ct \left( \|\nabla H\|_H \|\nabla H\|_{L^2}^2 + \|\rho^{1/2} \dot{u}\|_{L^2}^2 \right) \\ & + Ct^2 \left( \|\rho^{1/2} \dot{u}\|_{L^2}^2 + \|\nabla H\|_H \|\nabla H\|_{L^2}^2 \right) (\|\nabla u\|_{L^2}^2 + \|P\|_{L^2}^2 + \|H\|_{L^4}^4) \\ & + Ct^2 \|\nabla H\|_{L^2}^2 \|\nabla^2 H\|_{L^2}^2 + \tilde{C}t^2 \|P\|_{L^4}^4 \\ & \leq \hat{C}t^2 \|\Delta H\|_H \|\nabla H\|_{L^2}^2 + Ct \left( \|\rho^{1/2} \dot{u}\|_{L^2}^2 + \|\nabla H\|_H \|\nabla H\|_{L^2}^2 + \|\nabla^2 H\|_{L^2}^2 \right) \\ & \cdot (1 + t \|\nabla u\|_{L^2}^2 + t \|P\|_{L^2}^2 + t \|\nabla H\|_{L^2}^2 + t \|H\|_{L^4}^4) + \tilde{C}t^2 \|P\|_{L^4}^4 \end{aligned} \quad (3.57)$$

where in the second inequality one has used (2.9) and (2.1).

Choosing  $p = 3$  in (3.48), adding (3.48) multiplied by  $(\tilde{C} + 1)t^2$  to (3.57) lead to

$$\begin{aligned} & \left( t^2 \int \rho |\dot{u}|^2 dx + (\check{C}\hat{C} + 1)t^2 \|\nabla|\tilde{H}|^2\|_{L^2}^2 + \frac{2(2\mu + \lambda)(\tilde{C} + 1)}{3\gamma - 1} t^2 \|P\|_{L^3}^3 \right)_t \\ & + \mu t^2 \|\nabla \dot{u}\|_{L^2}^2 + (\check{C}\hat{C} + 1)t^2 \|\Delta|\tilde{H}|^2\|_{L^2}^2 + t^2 \|P\|_{L^4}^4 \\ & \leq \hat{C}t^2 \|\Delta H\|_H \|\nabla H\|_{L^2}^2 + Ct \left( \|\rho^{1/2} \dot{u}\|_{L^2}^2 + \|\nabla H\|_H \|\nabla H\|_{L^2}^2 + \|\nabla^2 H\|_{L^2}^2 + \|P\|_{L^3}^3 \right) \\ & \cdot (1 + t \|\nabla u\|_{L^2}^2 + t \|P\|_{L^2}^2 + t \|\nabla H\|_{L^2}^2 + t \|H\|_{L^4}^4) \end{aligned} \quad (3.58)$$

which combined with (3.54), (3.35), (3.36), (3.37), and (3.4) yields

$$\begin{aligned} & \sup_{\sigma(T) \leq t \leq T} t^2 \int (\rho |\dot{u}|^2 + |\nabla H|^2 |H|^2 + P^3) dx \\ & + \int_{\sigma(T)}^T t^2 (\|\nabla \dot{u}\|_{L^2}^2 + \|\Delta|\tilde{H}|^2\|_{L^2}^2 + \|\Delta H\|_H \|\nabla H\|_{L^2}^2 + \|P\|_{L^4}^4) dt \leq C(\bar{\rho})C_0. \end{aligned} \quad (3.59)$$

Finally, we claim that for  $m = 1, 2, \dots$ ,

$$\sup_{\sigma(T) \leq t \leq T} t^m \|P\|_{L^{m+1}}^{m+1} + \int_{\sigma(T)}^T t^m \|P\|_{L^{m+2}}^{m+2} dt \leq C(m, \bar{\rho})C_0, \quad (3.60)$$

which together with (2.9), (3.54), and (3.59) gives (3.46). Next, We shall prove (3.60) by induction. In fact, (3.54) shows that (3.60) holds for  $m = 1$ . Assume that (3.60) holds for  $m = n$ , that is,

$$\sup_{\sigma(T) \leq t \leq T} t^n \|P\|_{L^{n+1}}^{n+1} + \int_{\sigma(T)}^T t^n \|P\|_{L^{n+2}}^{n+2} dt \leq C(n, \bar{\rho})C_0. \quad (3.61)$$

Multiplying (3.48) where  $p = n + 2$  by  $t^{n+1}$ , one obtains after using (3.59)

$$\begin{aligned}
& \frac{2(2\mu + \lambda)}{(n+2)\gamma - 1} (t^{n+1} \|P\|_{L^{n+2}}^{n+2})_t + t^{n+1} \|P\|_{L^{n+3}}^{n+3} \\
& \leq C(n, \bar{\rho}) t^n \|P\|_{L^{n+2}}^{n+2} + C t^{n+1} \| |H|^2 \|_{L^{n+3}}^{n+3} \\
& \quad + C(n, \bar{\rho}) \left( t \|\rho^{1/2} \dot{u}\|_{L^2} + t \| |\nabla H| |H| \|_{L^2} \right)^{n+1} (\|\nabla u\|_{L^2}^2 + \|P\|_{L^2}^2 + \| |H|^2 \|_{L^2}^2) \\
& \leq C(n, \bar{\rho}) t^n \|P\|_{L^{n+2}}^{n+2} + C(n, \bar{\rho}) C_0 (\|\nabla u\|_{L^2}^2 + \|P\|_{L^2}^2 + \|\nabla H\|_{L^2}^2).
\end{aligned} \tag{3.62}$$

Integrating (3.62) over  $[\sigma(T), T]$  together with (3.61) and (3.38) shows that (3.60) holds for  $m = n + 1$ . By induction, we obtain (3.60) and finish the proof of Lemma 3.5.

Next, Lemma 2.4 combined with the following Lemma 3.6 which has been proved in [25, Lemma 3.5], will be useful to estimate the  $L^p$ -norm of  $\rho \dot{u}$  and obtain the uniform (in time) upper bound of the density for large time.

**Lemma 3.6** *Let  $(\rho, u, H)$  be a smooth solution of (1.1)-(1.4) on  $\mathbb{R}^2 \times (0, T]$  satisfying the assumptions in Theorem 1.1 and (3.4). Then for any  $\alpha > 0$ , there exists a positive constant  $N_1$  depending only on  $\alpha$ ,  $N_0$ , and  $M$  such that for all  $t \in (0, T]$ ,*

$$\int_{B_{N_1(1+t) \log^\alpha(1+t)}} \rho(x, t) dx \geq \frac{1}{4}. \tag{3.63}$$

Next, to obtain the upper bound of the density for small time, we still need the following lemmas about the short-time estimate about  $H$  and  $u$ .

**Lemma 3.7** *Let  $(\rho, u, H)$  be a smooth solution of (1.1)-(1.4) on  $\mathbb{R}^2 \times (0, T]$  satisfying (3.4) and the assumptions in Theorem 1.1. Then there exists a positive constant  $C$  depending only on  $\mu, \lambda, \nu, \gamma, a, \bar{\rho}, \beta, N_0$ , and  $M$  such that*

$$\sup_{0 \leq t \leq \sigma(T)} t^{1-\beta} (\|H\|_{L^2}^2 + \|\nabla H\|_{L^2}^2) + \int_0^{\sigma(T)} t^{1-\beta} (\|\nabla H\|_{L^2}^2 + \|\nabla^2 H\|_{L^2}^2) dt \leq C(M). \tag{3.64}$$

*Proof.* Multiplying (1.1)<sub>3</sub> by  $H$  and integrating by parts over  $\mathbb{R}^2$ , by (2.1) and Cauchy-Schwarz inequality, we have

$$\frac{1}{2} \frac{d}{dt} \|H\|_{L^2}^2 + \nu \|\nabla H\|_{L^2}^2 \leq \frac{\nu}{2} \|\nabla H\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 \|H\|_{L^2}^2 \tag{3.65}$$

which together with (3.6) and Gronwall's inequality implies that

$$\sup_{0 \leq t \leq T} \|H\|_{L^2}^2 + \int_0^T \|\nabla H\|_{L^2}^2 dt \leq C \|H_0\|_{L^2}^2. \tag{3.66}$$

Recall (3.19), we have by (2.1) and (2.2) that

$$\begin{aligned}
& \frac{d}{dt} \int |\nabla H|^2 dx + \int |\Delta H|^2 dx \leq C \int |\nabla u| |\nabla H|^2 dx + C \int |\nabla u| |H| |\Delta H| dx \\
& \leq C \|\nabla u\|_{L^2}^2 \|\nabla H\|_{L^2}^2 + C \|\nabla u\|_{L^2}^2 \|H\|_{L^\infty}^2 + \varepsilon \|\Delta H\|_{L^2}^2 \\
& \leq C \|\nabla u\|_{L^2}^2 \|\nabla H\|_{L^2}^2 + C \|\nabla u\|_{L^2}^2 \|H\|_{L^4} \|\nabla H\|_{L^4} + \varepsilon \|\Delta H\|_{L^2}^2 \\
& \leq C \|\nabla u\|_{L^2}^2 \|\nabla H\|_{L^2}^2 + C \|\nabla u\|_{L^2}^{8/3} \|H\|_{L^2}^{2/3} \|\nabla H\|_{L^2}^{4/3} + \varepsilon \|\Delta H\|_{L^2}^2 \\
& \leq C \|\nabla u\|_{L^2}^2 \|\nabla H\|_{L^2}^2 + C \|\nabla u\|_{L^2}^4 \|H\|_{L^2}^2 + \varepsilon \|\Delta H\|_{L^2}^2
\end{aligned} \tag{3.67}$$

which together with (3.65), and choosing  $\varepsilon$  suitably small, gives

$$\begin{aligned} \frac{d}{dt} (\|H\|_{L^2}^2 + \|\nabla H\|_{L^2}^2) + \|\nabla H\|_{L^2}^2 + \|\nabla^2 H\|_{L^2}^2 \\ \leq C (\|\nabla u\|_{L^2}^2 + \|\nabla u\|_{L^2}^4) (\|H\|_{L^2}^2 + \|\nabla H\|_{L^2}^2). \end{aligned} \quad (3.68)$$

Notice that, one has by (3.4) and (3.6) that

$$\begin{aligned} & \int_0^T (\|\nabla u\|_{L^2}^4 + \|\nabla H\|_{L^2}^4) dt \\ & \leq \int_0^{\sigma(T)} (\|\nabla u\|_{L^2}^4 + \|\nabla H\|_{L^2}^4) dt + \int_{\sigma(T)}^T \sigma (\|\nabla u\|_{L^2}^4 + \|\nabla H\|_{L^2}^4) dt \\ & \leq \sup_{0 \leq t \leq \sigma(T)} \left( \sigma^{(3-2\beta)/4} (\|\nabla u\|_{L^2}^2 + \|\nabla H\|_{L^2}^2) \right)^2 \int_0^{\sigma(T)} \sigma^{(2\beta-3)/2} dt \\ & \quad + \sup_{\sigma(T) \leq t \leq T} \sigma (\|\nabla u\|_{L^2}^2 + \|\nabla H\|_{L^2}^2) \int_{\sigma(T)}^T (\|\nabla u\|_{L^2}^2 + \|\nabla H\|_{L^2}^2) dt \\ & \leq CC_0^{2\delta_0} + CC_0^{3/2} \leq CC_0^{2\delta_0} \end{aligned} \quad (3.69)$$

since  $\beta \in (1/2, 1]$  and  $\delta_0 \in (0, 1/9]$ .

Applying Gronwall's inequality to (3.68), together with (3.6) and (3.69), we finally deduce that

$$\sup_{0 \leq t \leq T} (\|H\|_{L^2}^2 + \|\nabla H\|_{L^2}^2) + \int_0^T (\|\nabla H\|_{L^2}^2 + \|\nabla^2 H\|_{L^2}^2) dt \leq C \|H_0\|_{H^1}^2. \quad (3.70)$$

On the other hand, multiplying (3.68) by  $t$  and integrating it over  $(0, \sigma(T))$ , by (3.66), (3.6) and (3.69), it holds that

$$\sup_{0 \leq t \leq \sigma(T)} t (\|H\|_{L^2}^2 + \|\nabla H\|_{L^2}^2) + \int_0^{\sigma(T)} t (\|\nabla H\|_{L^2}^2 + \|\nabla^2 H\|_{L^2}^2) dt \leq C \|H_0\|_{H^0}^2. \quad (3.71)$$

Since the solution operator  $H_0 \mapsto H(\cdot, t)$  is linear, by the standard Stein-Weiss interpolation argument (see [2]), one can deduce from (3.70) (with  $T$  being replaced by  $\sigma(T)$ ) and (3.71) that for any  $\theta \in [\beta, 1]$ ,

$$\begin{aligned} \sup_{0 \leq t \leq \sigma(T)} t^{1-\theta} (\|H\|_{L^2}^2 + \|\nabla H\|_{L^2}^2) + \int_0^{\sigma(T)} t^{1-\theta} (\|\nabla H\|_{L^2}^2 + \|\nabla^2 H\|_{L^2}^2) dt \\ \leq C \|H_0\|_{H^\theta}^2 \leq C \|H_0\|_{H^\theta}^2 \end{aligned} \quad (3.72)$$

where in the last inequality one has used (3.66) and  $\|H_0\|_{L^2}^2 \leq M$ . Thus, we finish the proof of Lemma 3.7.

**Lemma 3.8** *Let  $(\rho, u, H)$  be a smooth solution of (1.1)-(1.4) on  $\mathbb{R}^2 \times (0, T]$  satisfying (3.4) and the assumptions in Theorem 1.1. Then there exists a positive constant  $C$  depending only on  $\mu, \lambda, \nu, \gamma, a, \bar{\rho}, \beta, N_0$ , and  $M$  such that*

$$\sup_{0 \leq t \leq \sigma(T)} t^{1-\beta} \|\nabla u\|_{L^2}^2 + \int_0^{\sigma(T)} t^{1-\beta} \int \rho |\dot{u}|^2 dx dt \leq C(\bar{\rho}, M). \quad (3.73)$$

*Proof.* First, we give out the following estimate

$$\sup_{0 \leq t \leq \sigma(T)} \int \bar{x}^a |H|^2 dx + \int_0^{\sigma(T)} \int \bar{x}^a |\nabla H|^2 dx dt \leq C(\|H_0 \bar{x}^{\frac{a}{2}}\|_{L^2}^2) \leq C(M), \quad (3.74)$$

which has been obtained in [31, Lemama 4.1], we give out the details here for completed. Multiplying (1.1)<sub>3</sub> by  $H \bar{x}^a$  and integration by parts yields

$$\begin{aligned} \frac{1}{2} \left( \int |H|^2 \bar{x}^a dx \right)_t + \nu \int |\nabla H|^2 \bar{x}^a dx &= \frac{\nu}{2} \int |H|^2 \Delta \bar{x}^a dx \\ &+ \int H \cdot \nabla u \cdot H \bar{x}^a dx - \frac{1}{2} \int \operatorname{div} u |H|^2 \bar{x}^a dx + \frac{1}{2} \int |H|^2 u \cdot \nabla \bar{x}^a dx = \sum_{i=1}^4 J_i. \end{aligned} \quad (3.75)$$

It easy to get from (2.1), (2.5) and Cauchy inequality that each term on the right-hand side of (3.75) can be estimated as follows

$$\begin{aligned} J_1 &\leq C \int |H|^2 \bar{x}^a \bar{x}^{-2} \log^{2(1+\eta_0)}(e + |x|^2) dx \leq C \int |H|^2 \bar{x}^a dx, \\ J_2 + J_3 &\leq C \int |\nabla u| |H|^2 \bar{x}^a dx \leq C \|\nabla u\|_{L^2} \|H \bar{x}^{\frac{a}{2}}\|_{L^4}^2 \\ &\leq C \|\nabla u\|_{L^2} \|H \bar{x}^{\frac{a}{2}}\|_{L^2} (\|\nabla H \bar{x}^{\frac{a}{2}}\|_{L^2} + \|H \nabla \bar{x}^{\frac{a}{2}}\|_{L^2}) \\ &\leq C(\|\nabla u\|_{L^2}^2 + 1) \|H \bar{x}^{\frac{a}{2}}\|_{L^2}^2 + \varepsilon \|\nabla H \bar{x}^{\frac{a}{2}}\|_{L^2}^2, \\ J_4 &\leq C \int |H| \bar{x}^{\frac{a}{2}} |H| \bar{x}^{\frac{a}{2}} |u| \bar{x}^{-\frac{1}{2}} \bar{x}^{-\frac{1}{2}} \log^{(1+\eta_0)}(e + |x|^2) dx \\ &\leq C \|H \bar{x}^{\frac{a}{2}}\|_{L^4} \|H \bar{x}^{\frac{a}{2}}\|_{L^2} \|u \bar{x}^{-\frac{1}{2}}\|_{L^4} \\ &\leq C \|H \bar{x}^{\frac{a}{2}}\|_{L^4}^2 + C \|H \bar{x}^{\frac{a}{2}}\|_{L^2}^2 \|u \bar{x}^{-\frac{1}{2}}\|_{L^4}^2 \\ &\leq C \left( 1 + \|\rho^{\frac{1}{2}} u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 \right) \|H \bar{x}^{\frac{a}{2}}\|_{L^2}^2 + \varepsilon \|\nabla H \bar{x}^{\frac{a}{2}}\|_{L^2}^2. \end{aligned} \quad (3.76)$$

Putting (3.76) into (3.75) and integrating the result over  $(0, \sigma(T))$ , we have by (3.6) and Gronwall's inequality that (3.74) holds.

Now, set

$$\nu \triangleq \min \left\{ \frac{\mu^{1/2}}{2(1 + 2\mu + \lambda)^{1/2}}, \frac{\beta}{1 - \beta} \right\} \in (0, 1/2].$$

If  $\beta \in (\frac{1}{2}, 1)$ , Sobolev's inequality implies

$$\begin{aligned} \int \rho_0 |u_0|^{2+\nu} dx &\leq \int \rho_0 |u_0|^2 dx + \int \rho_0 |u_0|^{2/(1-\beta)} dx \\ &\leq C(\bar{\rho}) + C(\bar{\rho}) \|u_0\|_{\dot{H}^\beta}^{2/(1-\beta)} \leq C(\bar{\rho}, M). \end{aligned} \quad (3.77)$$

For the case that  $\beta = 1$ , one obtains from (2.5) that

$$\int \rho_0 |u_0|^{2+\nu} dx \leq C(\bar{\rho}) \left( \int \rho_0 |u_0|^2 dx + \int |\nabla u_0|^2 dx \right)^{(2+\nu)/2} \leq C(\bar{\rho}, M). \quad (3.78)$$

Next, multiplying (1.1)<sub>2</sub> by  $(2 + \nu)|u|^\nu u$  and integrating the resulting equation over

$\mathbb{R}^2$  lead to

$$\begin{aligned}
& \frac{d}{dt} \int \rho |u|^{2+\nu} dx + (2+\nu) \int |u|^\nu (\mu |\nabla u|^2 + (\mu + \lambda)(\operatorname{div} u)^2) dx \\
& \leq (2+\nu) \nu \int (\mu + \lambda) |\operatorname{div} u| |u|^\nu |\nabla u| dx + C \int \rho^\gamma |u|^\nu |\nabla u| dx + C \int |H|^2 |u|^\nu |\nabla u| dx \\
& \leq \frac{2+\nu}{2} \int (\mu + \lambda)(\operatorname{div} u)^2 |u|^\nu dx + \frac{(2+\nu)\mu}{4} \int |u|^\nu |\nabla u|^2 dx \\
& \quad + C \int \rho |u|^{2+\nu} dx + C \int \rho^{(2+\nu)\gamma-\nu/2} dx + C \int |H|^4 |u|^\nu dx,
\end{aligned} \tag{3.79}$$

and the last term on the right-hand side of (3.79) can be estimated as follows

$$\begin{aligned}
& \int |H|^4 |u|^\nu dx \leq C \int (|H|^2)^{2-\nu} (|H|^2 |u|)^\nu dx \leq C \|H\|_{L^{\frac{4-2\nu}{1-\nu}}}^{\frac{4-2\nu}{1-\nu}} + C \int |H|^2 |u| dx \\
& \leq C \|H\|_{L^2}^2 \|\nabla H\|_{L^2}^{\frac{2}{1-\nu}} + C \int |H| |H| \bar{x}^{\frac{1}{2}} |u| \bar{x}^{-\frac{1}{2}} dx \\
& \leq C \|H\|_{L^2}^2 (1 + \|\nabla H\|_{L^2}^4) + C \|H\|_{L^4} \|H \bar{x}^{\frac{1}{2}}\|_{L^2} \|u \bar{x}^{-\frac{1}{2}}\|_{L^4} \\
& \leq C(M) (1 + \|\nabla H\|_{L^2}^4) + C \|H \bar{x}^{\frac{\alpha}{2}}\|_{L^2}^2 + C(M) \left( \|\rho^{\frac{1}{2}} u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 \right),
\end{aligned} \tag{3.80}$$

where  $\frac{2}{1-\nu} \in (2, 4]$  for  $\nu \in (0, \frac{1}{2}]$ . Then by Gronwall's inequality, together with (3.77), (3.78), (3.79) and (3.80), one easily shows that

$$\sup_{0 \leq t \leq \sigma(T)} \int \rho |u|^{2+\nu} dx \leq C(\bar{\rho}, M), \tag{3.81}$$

where we have used

$$\begin{aligned}
& \int_0^{\sigma(T)} \int |H|^4 |u|^\nu dx dt \leq C \sup_{0 \leq t \leq \sigma(T)} \|H \bar{x}^{\frac{\alpha}{2}}\|_{L^2}^2 \\
& \quad + C(M) \int_0^{\sigma(T)} (1 + \|\nabla u\|_{L^2}^2 + \|\nabla H\|_{L^2}^4) dt \\
& \leq C(\bar{\rho}, \|H_0 \bar{x}^{\frac{\alpha}{2}}\|_{L^2}) + C(\bar{\rho}, \|H_0\|_{\dot{H}^\beta}) \leq C(\bar{\rho}, M)
\end{aligned} \tag{3.82}$$

due to (3.80), (3.6), (3.74), (3.64) and (3.69).

Next, as in [14], for the linear differential operator  $L$  defined by

$$\begin{aligned}
(Lw)^j & \triangleq \rho w_t^j + \rho u \cdot \nabla w^j - (\mu \Delta w^j + (\mu + \lambda) \partial_j \operatorname{div} w) \\
& = \rho \dot{w}^j - (\mu \Delta w^j + (\mu + \lambda) \partial_j \operatorname{div} w), \quad j = 1, 2.
\end{aligned}$$

Let  $w_1, w_2$  and  $w_3$  be the solution to:

$$Lw_1 = 0, \quad w_1(x, 0) = w_{10}(x), \tag{3.83}$$

$$Lw_2 = -\nabla P(\rho), \quad w_2(x, 0) = 0, \tag{3.84}$$

$$Lw_3 = \frac{1}{2} (H \cdot \nabla B + B \cdot \nabla H) - \frac{1}{2} (\nabla B \cdot H + \nabla H \cdot B), \quad w_3(x, 0) = 0, \tag{3.85}$$

respectively, where  $B = (B^1, B^2)$  is the solution of (1.1)<sub>3</sub> with fixed  $u$  and initial data  $B_0(x)$ .

A straightforward energy estimate of (3.83) shows that:

$$\sup_{0 \leq t \leq \sigma(T)} \int \rho |w_1|^2 dx + \int_0^{\sigma(T)} \int |\nabla w_1|^2 dx dt \leq C(\bar{\rho}) \int |w_{10}|^2 dx. \quad (3.86)$$

Then, multiplying (3.83) by  $w_{1t}$  and integrating the resulting equality over  $\mathbb{R}^2$  yield that for  $t \in (0, \sigma(T)]$ ,

$$\begin{aligned} & \frac{1}{2} (\mu \|\nabla w_1\|_{L^2}^2 + (\mu + \lambda) \|\operatorname{div} w_1\|_{L^2}^2)_t + \int \rho |\dot{w}_1|^2 dx = \int \rho \dot{w}_1 (u \cdot \nabla w_1) dx \\ & \leq C(\bar{\rho}) \|\rho^{1/2} \dot{w}_1\|_{L^2} \|\rho^{1/(2+\nu)} u\|_{L^{2+\nu}} \|\nabla^2 w_1\|_{L^2}^{2/(2+\nu)} \|\nabla w_1\|_{L^2}^{\nu/(2+\nu)} \\ & \leq \frac{1}{2} \int \rho |\dot{w}_1|^2 dx + C(\bar{\rho}, M) \|\nabla w_1\|_{L^2}^2, \end{aligned} \quad (3.87)$$

where in the last inequality we have used (3.81) and the following simple fact:

$$\|\nabla^2 w_1\|_{L^2} \leq C \|\rho \dot{w}_1\|_{L^2}, \quad (3.88)$$

due to the standard  $L^2$ -estimate of the elliptic system (3.83). Then, by Gronwall's inequality together with (3.87) and (3.86) gives

$$\sup_{0 \leq t \leq \sigma(T)} \|\nabla w_1\|_{L^2}^2 + \int_0^{\sigma(T)} \int \rho |\dot{w}_1|^2 dx dt \leq C(\bar{\rho}, M) \|\nabla w_{10}\|_{L^2}^2, \quad (3.89)$$

and

$$\sup_{0 \leq t \leq \sigma(T)} t \|\nabla w_1\|_{L^2}^2 + \int_0^{\sigma(T)} t \int \rho |\dot{w}_1|^2 dx dt \leq C(\bar{\rho}, M) \|w_{10}\|_{L^2}^2. \quad (3.90)$$

Since the solution operator  $w_{10} \mapsto w_1(\cdot, t)$  is linear, by the standard Stein-Weiss interpolation argument (see [2]), one can deduce from (3.89) and (3.90) that for any  $\theta \in [\beta, 1]$ ,

$$\sup_{0 \leq t \leq \sigma(T)} t^{1-\theta} \|\nabla w_1\|_{L^2}^2 + \int_0^{\sigma(T)} t^{1-\theta} \int \rho |\dot{w}_1|^2 dx dt \leq C(\bar{\rho}, M) \|w_{10}\|_{H^\theta}^2, \quad (3.91)$$

with a uniform constant  $C$  independent of  $\theta$ .

Now, we estimate  $w_2$ . It follows from a similar way as for the proof of (2.7) and (2.9) that

$$\|\nabla((2\mu + \lambda)\operatorname{div} w_2 - P)\|_{L^2} + \|\nabla(\nabla^\perp \cdot w_2)\|_{L^2} \leq C \|\rho \dot{w}_2\|_{L^2}, \quad (3.92)$$

and that for  $p \geq 2$ ,

$$\begin{aligned} \|\nabla w_2\|_{L^p} & \leq C(\|(2\mu + \lambda)\operatorname{div} w_2 - P\|_{L^p} + C\|P\|_{L^p} + \|\nabla^\perp \cdot w_2\|_{L^p}) \\ & \leq \delta \|\rho \dot{w}_2\|_{L^2} + C(\bar{\rho}, p, \delta) \|\nabla w_2\|_{L^2} + C(\bar{\rho}, p, \delta) C_0^{1/p}. \end{aligned} \quad (3.93)$$

Multiplying (3.84) by  $w_{2t}$  and integrating the resulting equation over  $\mathbb{R}^2$  yield that for  $t \in (0, \sigma(T)]$ ,

$$\begin{aligned} & \frac{1}{2} \left( \mu \|\nabla w_2\|_{L^2}^2 + (\mu + \lambda) \|\operatorname{div} w_2\|_{L^2}^2 - 2 \int P \operatorname{div} w_2 dx \right)_t + \int \rho |\dot{w}_2|^2 dx \\ & = \int \rho \dot{w}_2 (u \cdot \nabla w_2) dx - \int P_t \operatorname{div} w_2 dx \\ & \leq C(\bar{\rho}) \|\rho^{1/2} \dot{w}_2\|_{L^2} \|\rho^{1/(2+\nu)} u\|_{L^{2+\nu}} \|\nabla w_2\|_{L^{2(2+\nu)/\nu}} - \int P_t \operatorname{div} w_2 dx \\ & \leq C(\bar{\rho}, M) \delta \|\rho^{1/2} \dot{w}_2\|_{L^2}^2 + C(\delta, \bar{\rho}, M) (\|\nabla w_2\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + 1), \end{aligned} \quad (3.94)$$

where in the last inequality we have used (3.93), (3.81), and the following simple fact:

$$\begin{aligned}
-\int P_t \operatorname{div} w_2 dx &= -\frac{1}{2\mu + \lambda} \int Pu \cdot \nabla((2\mu + \lambda) \operatorname{div} w_2 - P) dx \\
&\quad + \frac{1}{2(2\mu + \lambda)} \int P^2 \operatorname{div} u dx + (\gamma - 1) \int P \operatorname{div} u \operatorname{div} w_2 dx \\
&\leq C \|Pu\|_{L^2} \|\rho \dot{w}_2\|_{L^2} + C \|P^2\|_{L^2} \|\nabla u\|_{L^2} + C \|\nabla u\|_{L^2}^2 + C \|\nabla w_2\|_{L^2}^2 \\
&\leq \delta \|\rho^{1/2} \dot{w}_2\|_{L^2}^2 + C(\delta, \bar{\rho}) (\|\nabla u\|_{L^2}^2 + \|\nabla w_2\|_{L^2}^2 + 1),
\end{aligned}$$

due to (3.13) and (3.92). The combination of Gronwall's inequality with (3.94) gives

$$\sup_{0 \leq t \leq \sigma(T)} \|\nabla w_2\|_{L^2}^2 + \int_0^{\sigma(T)} \int \rho |\dot{w}_2|^2 dx dt \leq C(\bar{\rho}, M). \quad (3.95)$$

Now, we estimate  $w_3$ . Multiplying (3.85) by  $\dot{w}_3 = w_{3t} + u \cdot \nabla w_3$  and integrating by parts over  $\mathbb{R}^2$  lead to

$$\begin{aligned}
\frac{\mu}{2} \frac{d}{dt} \int |\nabla w_3|^2 dx + \int \rho |\dot{w}_3|^2 dx &\leq C \int |\nabla u| |\nabla w_3|^2 dx \\
+ \frac{1}{2} \int (H \cdot \nabla B + B \cdot \nabla H) \cdot \dot{w}_3 dx &- \frac{1}{2} \int \nabla(B \cdot H) \cdot \dot{w}_3 dx.
\end{aligned} \quad (3.96)$$

Next, we will estimate the three terms on the left of (3.96). For the second term on the left of (3.96), some straightforward calculations yield that

$$\begin{aligned}
&\frac{1}{2} \int (H \cdot \nabla B + B \cdot \nabla H) \cdot \dot{w}_3 dx \\
&= \frac{1}{2} \int (H \cdot \nabla B + B \cdot \nabla H) \cdot w_{3t} dx + \frac{1}{2} \int (H \cdot \nabla B + B \cdot \nabla H) \cdot (u \cdot \nabla w_3) dx \\
&= -\frac{1}{2} \frac{d}{dt} \left( \int H \cdot \nabla w_3 \cdot B dx + \int B \cdot \nabla w_3 \cdot H dx \right) \\
&\quad + \frac{1}{2} \int H_t \cdot \nabla w_3 \cdot B dx + \frac{1}{2} \int H \cdot \nabla w_3 \cdot B_t dx + \frac{1}{2} \int H \cdot \nabla B \cdot (u \cdot \nabla w_3) dx \\
&\quad + \frac{1}{2} \int B_t \cdot \nabla w_3 \cdot H dx + \frac{1}{2} \int B \cdot \nabla w_3 \cdot H_t dx + \frac{1}{2} \int B \cdot \nabla H \cdot (u \cdot \nabla w_3) dx \\
&= \frac{d}{dt} \hat{I}_0 + \hat{I}_1 + \hat{I}_2 + \hat{I}_3 + \hat{I}_4 + \hat{I}_5 + \hat{I}_6.
\end{aligned} \quad (3.97)$$

Denote

$$\bar{D}(u, x) \triangleq x \cdot \nabla u - x \operatorname{div} u + \nu \Delta x, \quad x = (x^1, x^2) \in \mathbb{R}^2. \quad (3.98)$$

It is easy to deduce from (1.1)<sub>3</sub> and integrating by parts that

$$\begin{aligned}
\hat{I}_1 &= -\frac{1}{2} \int (u \cdot \nabla H) \cdot \nabla w_3 \cdot B dx + \frac{1}{2} \int \bar{D}(u, H) \cdot \nabla w_3 \cdot B dx \\
&= \frac{1}{2} \int \operatorname{div} u H \cdot \nabla w_3 \cdot B dx + \frac{1}{2} \int u^i H^j (\partial_i \partial_j w_3^k) B^k dx \\
&\quad + \frac{1}{2} \int (H \cdot \nabla w_3) \cdot (u \cdot \nabla B) dx + \frac{1}{2} \int \bar{D}(u, H) \cdot \nabla w_3 \cdot B dx, \\
\hat{I}_2 &= -\frac{1}{2} \int (H \cdot \nabla w_3) \cdot (u \cdot \nabla B) dx + \frac{1}{2} \int (H \cdot \nabla w_3) \cdot \bar{D}(u, B) dx, \\
\hat{I}_3 &= -\frac{1}{2} \int H^k B^j \partial_k u^i \partial_i w_3^j dx - \frac{1}{2} \int H^k B^j u^i (\partial_k \partial_i w_3^j) dx,
\end{aligned} \quad (3.99)$$

which implies that

$$\begin{aligned}\hat{I}_1 + \hat{I}_2 + \hat{I}_3 &= \frac{1}{2} \int \operatorname{div} u H \cdot \nabla w_3 \cdot B dx - \frac{1}{2} \int H \cdot \nabla u \cdot \nabla w_3 \cdot B dx \\ &\quad + \frac{1}{2} \int \bar{D}(u, H) \cdot \nabla w_3 \cdot B dx + \frac{1}{2} \int (H \cdot \nabla w_3) \cdot \bar{D}(u, B) dx.\end{aligned}\quad (3.100)$$

Similarly as (3.99)-(3.100), we also have

$$\begin{aligned}\hat{I}_4 + \hat{I}_5 + \hat{I}_6 &= \frac{1}{2} \int \operatorname{div} u B \cdot \nabla w_3 \cdot H dx - \frac{1}{2} \int B \cdot \nabla u \cdot \nabla w_3 \cdot H dx \\ &\quad + \frac{1}{2} \int \bar{D}(u, B) \cdot \nabla w_3 \cdot H dx + \frac{1}{2} \int (B \cdot \nabla w_3) \cdot \bar{D}(u, H) dx.\end{aligned}\quad (3.101)$$

Following the same arguments as (3.97)-(3.100), it holds that

$$\begin{aligned}-\frac{1}{2} \int \nabla(B \cdot H) \cdot \dot{w}_3 dx &= \frac{1}{2} \frac{d}{dt} \int (B \cdot H) \operatorname{div} w_3 dx - \frac{1}{2} \int (B \cdot H_t) \operatorname{div} w_3 dx \\ &\quad - \frac{1}{2} \int (B_t \cdot H) \operatorname{div} w_3 dx + \frac{1}{2} \int (B \cdot H) \operatorname{div}(u \cdot \nabla w_3) dx \\ &= \frac{d}{dt} \check{I}_0 - \frac{1}{2} \int \operatorname{div} u (H \cdot B) \operatorname{div} w_3 dx - \frac{1}{2} \int \bar{D}(u \cdot H) \cdot B \operatorname{div} w_3 dx \\ &\quad - \frac{1}{2} \int \bar{D}(u, B) \cdot H \operatorname{div} w_3 dx + \frac{1}{2} \int (H \cdot B) (\nabla u \cdot \nabla w_3) dx.\end{aligned}\quad (3.102)$$

Putting (3.100), (3.101) and (3.102) into (3.96), we have by (3.98) that

$$\begin{aligned}\frac{\mu}{2} \frac{d}{dt} \int |\nabla w_3|^2 dx + \int \rho |\dot{w}_3|^2 dx &\leq \frac{d}{dt} (\hat{I}_0 + \check{I}_0) + C \int |\nabla u| |\nabla w_3|^2 dx \\ &\quad + C \int |H| |B| |\nabla u| |\nabla w_3| dx + C \int |\nabla H| |B| |\nabla^2 w_3| dx + C \int |H| |\nabla B| |\nabla^2 w_3| dx \\ &\quad + C \int |\nabla H| |\nabla B| |\nabla w_3| dx \triangleq \frac{d}{dt} (\hat{I}_0 + \check{I}_0) + \tilde{I}_1 + \tilde{I}_2 + \tilde{I}_3 + \tilde{I}_4 + \tilde{I}_5,\end{aligned}\quad (3.103)$$

where

$$\begin{aligned}\hat{I}_0 + \check{I}_0 &\leq \int |H| |B| |\nabla w_3| dx \leq \frac{\mu}{4} \|\nabla w_3\|_{L^2}^2 + C \|H\|_{L^4}^2 \|B\|_{L^4}^2 \\ &\leq \frac{\mu}{4} \|\nabla w_3\|_{L^2}^2 + C(M) (\|B\|_{L^2}^2 + \|\nabla B\|_{L^2}^2),\end{aligned}\quad (3.104)$$

$$\tilde{I}_1 \leq \|\nabla u\|_{L^2} \|\nabla w_3\|_{L^4}^2 \leq \varepsilon \|\nabla^2 w_3\|_{L^2}^2 + C \|\nabla u\|_{L^2}^2 \|\nabla w_3\|_{L^2}^2, \quad (3.105)$$

$$\begin{aligned}\tilde{I}_2 &\leq C \int |\nabla u| |\nabla w_3|^2 dx + C \int |H|^2 |B|^2 |\nabla u| dx \\ &\leq C \tilde{I}_1 + C \| |H| |B| \|_{L^4}^2 \|\nabla u\|_{L^2} \\ &\leq C \tilde{I}_1 + C \|\nabla(|H| |B|)\|_{L^2}^2 + C \| |H| |B| \|_{L^2}^2 \|\nabla u\|_{L^2}^2 \\ &\leq C \tilde{I}_1 + C \| |B| |\nabla H| \|_{L^2}^2 + C \| |H| |\nabla B| \|_{L^2}^2 + C \|H\|_{L^4}^2 \|B\|_{L^4}^2 \|\nabla u\|_{L^2}^2 \\ &\leq \varepsilon \|\nabla^2 w_3\|_{L^2}^2 + C \|\nabla u\|_{L^2}^2 \|\nabla w_3\|_{L^2}^2 + C \| |B| |\nabla H| \|_{L^2}^2 + C \| |H| |\nabla B| \|_{L^2}^2 \\ &\quad + C(M) (\|B\|_{L^2}^2 + \|\nabla B\|_{L^2}^2) \|\nabla u\|_{L^2}^2,\end{aligned}\quad (3.106)$$

$$\tilde{I}_3 + \tilde{I}_4 \leq \varepsilon \|\nabla^2 w_3\|_{L^2}^2 + C \| |B| |\nabla H| \|_{L^2}^2 + C \| |H| |\nabla B| \|_{L^2}^2, \quad (3.107)$$



$$\begin{aligned}
\tilde{I}_5 &\leq C\|\nabla H\|_{L^2}\|\nabla B\|_{L^4}\|\nabla w_3\|_{L^4} \\
&\leq C\|\nabla H\|_{L^2}\|\nabla B\|_{L^2}^{1/2}\|\nabla^2 B\|_{L^2}^{1/2}\|\nabla w_3\|_{L^2}^{1/2}\|\nabla^2 w_3\|_{L^2}^{1/2} \\
&\leq \varepsilon\|\nabla^2 w_3\|_{L^2}^2 + C\|\nabla H\|_{L^2}^{4/3}\|\nabla B\|_{L^2}^{2/3}\|\nabla^2 B\|_{L^2}^{2/3}\|\nabla w_3\|_{L^2}^{2/3} \\
&\leq \varepsilon\|\nabla^2 w_3\|_{L^2}^2 + C\|\nabla H\|_{L^2}^2\|\nabla w_3\|_{L^2}^2 + C\|\nabla H\|_{L^2}^2\|\nabla B\|_{L^2}^2 + C\|\nabla^2 B\|_{L^2}^2.
\end{aligned} \tag{3.108}$$

Note that

$$\|\nabla^2 w_3\|_{L^2}^2 \leq C\|\rho^{1/2}\dot{w}_3\|_{L^2}^2 + C\|H\|\nabla B\|_{L^2}^2 + C\|B\|\nabla H\|_{L^2}^2 \tag{3.109}$$

which is deduced from (3.85) directly, then putting (3.105)-(3.108) into (3.103) and choosing  $\varepsilon$  suitably small, we have

$$\begin{aligned}
&\frac{\mu}{2}\frac{d}{dt}\int|\nabla w_3|^2dx + \int\rho|\dot{w}_3|^2dx \\
&\leq \frac{d}{dt}(\hat{I}_0 + \check{I}_0) + C(\|\nabla u\|_{L^2}^2 + \|\nabla H\|_{L^2}^2)\|\nabla w_3\|_{L^2}^2 \\
&\quad + C\|B\|\nabla H\|_{L^2}^2 + C\|H\|\nabla B\|_{L^2}^2 + C\|\nabla^2 B\|_{L^2}^2 \\
&\quad + C(M)(\|\nabla H\|_{L^2}^2 + \|\nabla u\|_{L^2}^2)(\|B\|_{L^2}^2 + \|\nabla B\|_{L^2}^2).
\end{aligned} \tag{3.110}$$

Furthermore, multiplying (1.1)<sub>3</sub> by  $H|B|^2$ , and multiplying (1.1)<sub>3</sub> (replaced the  $H$  in (1.1)<sub>3</sub> by  $B$ ) by  $B|H|^2$ , then adding the upper two resultant equations together and integrating it by parts over  $\mathbb{R}^2$ , we have

$$\begin{aligned}
&\frac{d}{dt}(\|B\|H\|_{L^2}^2 + \nu\|\nabla H\|B\|_{L^2}^2 + \nu\|\nabla B\|H\|_{L^2}^2) \\
&\leq C\int|\nabla H\|H\|\nabla B\|B|dx + C\int|\nabla u\|H|^2|B|^2dx \\
&\leq \|\nabla H\|B\|_{L^2}\|\nabla B\|_{L^4}\|H\|_{L^4} + C\|\nabla u\|_{L^2}\|H\|B\|_{L^2}\|\nabla(|H|B|)\|_{L^2} \\
&\leq \varepsilon(\|\nabla H\|B\|_{L^2}^2 + \|\nabla B\|H\|_{L^2}^2) + C\|\nabla B\|_{L^4}^2\|H\|_{L^4}^2 + C\|\nabla u\|_{L^2}^2\|H\|B\|_{L^2}^2 \\
&\leq C(M)(\|\nabla B\|_{L^2}^2 + \|\Delta B\|_{L^2}^2) + C\|\nabla u\|_{L^2}^2\|H\|B\|_{L^2}^2
\end{aligned} \tag{3.111}$$

where in the last inequality one has choosing  $\varepsilon$  suitably small. Recall  $B$  satisfies (3.70) and (3.71), applying Gronwall's inequality to (3.111) and together with (3.6), it holds that

$$\begin{aligned}
&\sup_{0\leq t\leq\sigma(T)}(\|B\|H\|_{L^2}^2) + \int_0^{\sigma(T)}(\|\nabla H\|B\|_{L^2}^2 + \|\nabla B\|H\|_{L^2}^2)dt \\
&\leq C\|H_0\|B_0\|_{L^2}^2 + C(M)\int_0^{\sigma(T)}(\|\nabla B\|_{L^2}^2 + \|\Delta B\|_{L^2}^2)dt \\
&\leq C\|H_0\|_{L^4}^2(\|B_0\|_{L^2}^2 + \|\nabla B_0\|_{L^2}^2) + C(M)\|B_0\|_{H^1}^2 \leq C(M)\|B_0\|_{H^1}^2
\end{aligned} \tag{3.112}$$

and

$$\begin{aligned}
&\sup_{0\leq t\leq\sigma(T)}(t\|B\|H\|_{L^2}^2) + \int_0^{\sigma(T)}t(\|\nabla H\|B\|_{L^2}^2 + \|\nabla B\|H\|_{L^2}^2)dt \\
&\leq C\int\|B\|H\|_{L^2}^2dt + C(M)\int_0^{\sigma(T)}t(\|\nabla B\|_{L^2}^2 + \|\Delta B\|_{L^2}^2)dt \\
&\leq C\int\|H\|_{L^4}^2(\|B\|_{L^2}^2 + \|\nabla B\|_{L^2}^2)dt + C(M)\|B_0\|_{L^2}^2 \leq C(M)\|B_0\|_{L^2}^2
\end{aligned} \tag{3.113}$$

Now, applying Gronwall's inequality to (3.110), together with (3.70), (3.71), (3.104), (3.112) and (3.113) gives

$$\sup_{0 \leq t \leq \sigma(T)} \|\nabla w_3\|_{L^2}^2 + \int_0^{\sigma(T)} \int \rho |\dot{w}_3|^2 dx dt \leq C(M) \|B_0\|_{H^1}^2, \quad (3.114)$$

$$\sup_{0 \leq t \leq \sigma(T)} t \|\nabla w_3\|_{L^2}^2 + \int_0^{\sigma(T)} t \int \rho |\dot{w}_3|^2 dx dt \leq C(M) \|B_0\|_{H^0}^2. \quad (3.115)$$

Since the solution operator  $B_0 \mapsto B$  and  $B \mapsto w_3$  is linear, so that  $B_0 \mapsto w_3$  is also linear, we thus conclude from (3.114) and (3.115) in a manner similar to the derivation of (3.72) that

$$\sup_{0 \leq t \leq \sigma(T)} t^{1-\theta} \|\nabla w_3\|_{L^2}^2 + \int_0^{\sigma(T)} t^{1-\theta} \int \rho |\dot{w}_3|^2 dx dt \leq C(M) \|H_0\|_{H^\theta}^2 \quad (3.116)$$

Finally, choosing  $w_{10} = u_0$  and  $B_0 \mapsto H_0$ , so that  $w_1 + w_2 + w_3 = u$  and  $B = H$ , we immediately obtain (3.73) from (3.91), (3.95) and (3.116). Thus, we finish the proof of Lemma 3.8.

We now proceed to derive a uniform (in time) upper bound for the density, which turns out to be the key to obtain all the higher order estimates. We will use an approach motivated by Li J. and Xin Z.P. ( see [25] or [19]).

**Lemma 3.9** *There exists a positive constant  $\varepsilon_0 = \varepsilon_0(\bar{\rho}, M)$  depending on  $\mu, \lambda, \nu, \gamma, a, \bar{\rho}, \beta, N_0$ , and  $M$  such that, if  $(\rho, u, H)$  is a smooth solution of (1.1)-(1.4) on  $\mathbb{R}^2 \times (0, T]$  satisfying (3.4) and the assumptions in Theorem 1.1, then*

$$\sup_{0 \leq t \leq T} \|\rho(t)\|_{L^\infty} \leq \frac{7\bar{\rho}}{4}, \quad (3.117)$$

provided  $C_0 \leq \varepsilon_0$ .

*Proof.* First, we rewrite the equation of the mass conservation (1.1)<sub>1</sub> as

$$D_t \rho = g(\rho) + b'(t), \quad (3.118)$$

where

$$D_t \rho \triangleq \rho_t + u \cdot \nabla \rho, \quad g(\rho) \triangleq -\frac{\rho^{\gamma+1}}{2\mu + \lambda}, \quad b(t) \triangleq -\frac{1}{2\mu + \lambda} \int_0^t \rho \left( F + \frac{1}{2} |H|^2 \right) dt.$$

Notice that we have by Gagliardo-Nirenberg and Hölder inequality that

$$\begin{aligned} \| |H| |\nabla H| \|_{L^p} &\leq \|H\|_{L^{2q}} \|\nabla H\|_{L^{2q}} \leq C(p) \|H\|_{L^2}^{\frac{1}{p}} \|\nabla H\|_{L^2} \|\Delta H\|_{L^2}^{\frac{p-1}{p}} \\ &\leq C(p) \|\nabla H\|_{L^2} (\|H\|_{L^2} + \|\Delta H\|_{L^2}) \end{aligned} \quad (3.119)$$

which together with (2.7), (3.63), and (2.5) implies that for  $t > 0$  and  $p \in [2, \infty)$ ,

$$\begin{aligned} \left\| \nabla \left( F + \frac{1}{2} |H|^2 \right) \right\|_{L^p} &\leq C \|\nabla F\|_{L^p} + C \|\nabla |H|^2\|_{L^p} \\ &\leq C(p) \|\rho \dot{u}\|_{L^p} + C(p) \|H\| \|\nabla H\|_{L^p} \\ &\leq C(p, \bar{\rho}, M) (1+t)^5 \left( \|\rho^{1/2} \dot{u}\|_{L^2} + \|\nabla \dot{u}\|_{L^2} \right) \\ &\quad + C(p) \|\nabla H\|_{L^2} (\|H\|_{L^2} + \|\Delta H\|_{L^2}) \end{aligned} \quad (3.120)$$

where in the last inequality we have used (3.119) and following estimate

$$\sup_{0 \leq s \leq t} \int \rho(1 + |x|^2)^{1/2} dx \leq C(M)(1 + t) \quad (3.121)$$

which is obtained after multiplying (1.1)<sub>1</sub> by  $(1 + |x|^2)^{1/2}$  and integrating the resulting equality over  $\mathbb{R}^2$  by parts.

Choosing  $q = 2$  in the Gagliardo-Nirenberg inequality (2.2), together with (3.120), we have for  $r \triangleq 4 + 4/\beta$ ,  $\delta_0 \triangleq (2r + (1 - \beta)(r - 2))/(3r - 4) \in (0, 1)$ , that

$$\begin{aligned} |b(\sigma(T))| &\leq C(\bar{\rho}) \int_0^{\sigma(T)} \sigma^{-\frac{2r+(1-\beta)(r-2)}{4(r-1)}} \left( \sigma^{1-\beta} (\|F\|_{L^2}^2 + \| |H|^2 \|_{L^2}^2) \right)^{\frac{r-2}{4(r-1)}} \\ &\quad \cdot \left( \sigma^2 \|\nabla(F + |H|^2)\|_{L^r}^2 \right)^{\frac{r}{4(r-1)}} dt \\ &\leq C(\bar{\rho}, M) \int_0^{\sigma(T)} \sigma^{-\frac{2r+(1-\beta)(r-2)}{4(r-1)}} \left( \sigma^2 \|\nabla(F + |H|^2)\|_{L^r}^2 \right)^{\frac{r}{4(r-1)}} dt \\ &\leq C(\bar{\rho}, M) \left( \int_0^{\sigma(T)} \sigma^{-\delta_0} dt \right)^{\frac{3r-4}{4(r-1)}} \left( \int_0^{\sigma(T)} \sigma^2 \|\nabla(F + |H|^2)\|_{L^r}^2 dt \right)^{\frac{r}{4(r-1)}} \\ &\leq C(\bar{\rho}, M) C_0^{\frac{r}{4(r-1)}} \end{aligned} \quad (3.122)$$

where in the second inequality one has used (3.64) and (3.73), and in the last inequality one has used the following estimate

$$\begin{aligned} \int_0^{\sigma(T)} \sigma^2 \|\nabla(F + |H|^2)\|_{L^r}^2 dt &\leq C \int_0^{\sigma(T)} \sigma^2 \left( \|\rho^{1/2} \dot{u}\|_{L^2}^2 + \|\nabla \dot{u}\|_{L^2}^2 \right) dt \\ &\quad + C \int_0^{\sigma(T)} \sigma^2 \|\nabla H\|_{L^2}^2 (\|H\|_{L^2}^2 + \|\triangle H\|_{L^2}^2) dt \\ &\leq CC_0 + C \left( \sup_{t \in [0, \sigma(T)]} (\sigma \|\nabla H\|_{L^2}^2) \right) \int_0^{\sigma(T)} (\|H\|_{L^2}^2 + \sigma \|\triangle H\|_{L^2}^2) dt \\ &\leq CC_0 \end{aligned} \quad (3.123)$$

owing to (3.6) and (3.38). Hence, (3.122) combined with (3.118) yields that

$$\sup_{t \in [0, \sigma(T)]} \|\rho\|_{L^\infty} \leq \bar{\rho} + C(\bar{\rho}, M) C_0^{1/4} \leq \frac{3\bar{\rho}}{2}, \quad (3.124)$$

provided

$$C_0 \leq \varepsilon_1 \triangleq \min\{1, (\bar{\rho}/(2C(\bar{\rho}, M)))^4\}.$$

Next, it follows from (2.7) and (3.46) that for  $t \in [\sigma(T), T]$ ,

$$\begin{aligned} \|F + |H|^2\|_{H^1} &\leq C (\|\nabla u\|_{L^2} + \|P\|_{L^2} + \| |H|^2 \|_{L^2}) \\ &\quad + C (\|\rho \dot{u}\|_{L^2} + \| |H| \|\nabla H\|_{L^2} + \|\nabla |H|^2\|_{L^2}) \\ &\leq C(\bar{\rho}) C_0^{1/2} t^{-1/2}, \end{aligned} \quad (3.125)$$

which together with (2.2) and (3.120) shows

$$\begin{aligned}
\int_{\sigma(T)}^T \|F + |H|^2\|_{L^\infty}^4 dt &\leq C \int_{\sigma(T)}^T \|F + |H|^2\|_{L^{72}}^{\frac{35}{9}} \|\nabla F + \nabla |H|^2\|_{L^{72}}^{\frac{1}{9}} dt \\
&\leq C(\bar{\rho}, M) C_0^{35/18} \int_{\sigma(T)}^T \left[ t^{-\frac{25}{18}} (\|\rho^{1/2} \dot{u}\|_{L^2} + \|\nabla \dot{u}\|_{L^2})^{\frac{1}{9}} \right. \\
&\quad \left. + t^{-\frac{35}{18}} (\|\nabla H\|_{L^2} (\|H\|_{L^2} + \|\Delta H\|_{L^2}))^{\frac{1}{9}} \right] dt \\
&\leq C(\bar{\rho}, M) C_0^{35/18},
\end{aligned} \tag{3.126}$$

where in the last inequality, one has used (3.4) and (3.6). This shows that for all  $\sigma(T) \leq t_1 \leq t_2 \leq T$ ,

$$\begin{aligned}
|b(t_2) - b(t_1)| &\leq C(\bar{\rho}) \int_{t_1}^{t_2} \|F + |H|^2\|_{L^\infty} dt \\
&\leq \frac{1}{2\mu + \lambda} (t_2 - t_1) + C(\bar{\rho}, M) \int_{\sigma(T)}^T \|F + |H|^2\|_{L^\infty}^4 dt \\
&\leq \frac{1}{2\mu + \lambda} (t_2 - t_1) + C(\bar{\rho}, M) C_0^{35/18},
\end{aligned}$$

which implies that one can choose  $N_1$  and  $N_0$  in (2.11) as:

$$N_1 = \frac{1}{2\mu + \lambda}, \quad N_0 = C(\bar{\rho}, M) C_0^{35/18}.$$

Hence, we set  $\bar{\zeta} = 1$  in (2.12) since for all  $\zeta \geq 1$ ,

$$g(\zeta) = -\frac{\zeta^{\gamma+1}}{2\mu + \lambda} \leq -N_1 = -\frac{1}{2\mu + \lambda}.$$

Lemma 2.6 and (3.124) thus lead to

$$\sup_{t \in [\sigma(T), T]} \|\rho\|_{L^\infty} \leq \frac{3\bar{\rho}}{2} + N_0 \leq \frac{7\bar{\rho}}{4}, \tag{3.127}$$

provided

$$C_0 \leq \varepsilon_0 \triangleq \min\{\varepsilon_1, \varepsilon_2\}, \quad \text{for } \varepsilon_2 \triangleq \left( \frac{\bar{\rho}}{4C(\bar{\rho}, M)} \right)^{18/35}.$$

The combination of (3.124) with (3.127) completes the proof of Lemma 3.9.

With Lemma 3.4 and Lemma 3.9 at hand, we are now in a position to prove Proposition 3.1.

*Proof of Proposition 3.1.* It follows from (3.38) that

$$A_1(T) + A_2(T) + \int_0^T \sigma \|P\|_{L^2}^2 dt \leq C_0^{1/2} \tag{3.128}$$

provided

$$C_0 \leq \varepsilon_3 \triangleq (C(\bar{\rho}))^{-2}.$$

It remains to estimate  $A_3(\sigma(T))$ . Indeed, using (3.128), (3.64) and (3.73), it holds that

$$\begin{aligned}
A_3(T) &\leq \sup_{0 \leq t \leq T} \left( \sigma^{1-\beta} \|\nabla u\|_{L^2}^2 \right)^{(2\beta+1)/(4\beta)} \sup_{0 \leq t \leq T} \left( \sigma \|\nabla u\|_{L^2}^2 \right)^{(2\beta-1)/(4\beta)} \\
&\quad + \sup_{0 \leq t \leq T} \left( \sigma^{1-\beta} \|\nabla H\|_{L^2}^2 \right)^{(2\beta+1)/(4\beta)} \sup_{0 \leq t \leq T} \left( \sigma \|\nabla H\|_{L^2}^2 \right)^{(2\beta-1)/(4\beta)} \\
&\quad + \left( \int_0^T \sigma^{1-\beta} \|\rho^{1/2} \dot{u}\|_{L^2}^2 dt \right)^{(2\beta+1)/(4\beta)} \left( \int_0^T \sigma \|\rho^{1/2} \dot{u}\|_{L^2}^2 dt \right)^{(2\beta-1)/(4\beta)} \\
&\quad + \left( \int_0^T \sigma^{1-\beta} \|\nabla^2 H\|_{L^2}^2 dt \right)^{(2\beta+1)/(4\beta)} \left( \int_0^T \sigma \|\nabla^2 H\|_{L^2}^2 dt \right)^{(2\beta-1)/(4\beta)} \\
&\leq C(\bar{\rho}, M) A_1^{(2\beta-1)/(4\beta)}(T) \leq C(\bar{\rho}, M) C_0^{(2\beta-1)/(8\beta)} \leq C_0^{\delta_0}
\end{aligned} \tag{3.129}$$

provided

$$C_0 \leq \varepsilon_4 \triangleq C(\bar{\rho}, M)^{(-72\beta)/(2\beta-1)}.$$

Letting  $\varepsilon \triangleq \min\{\varepsilon_0, \varepsilon_3, \varepsilon_4\}$ , we obtain (3.5) directly from (3.117), (3.128) and (3.129) and finish the proof of Proposition 3.1.

## 4 A priori estimates (II): higher order estimates

Form now on, for smooth initial data  $(\rho_0, u_0, H_0)$  satisfying (1.8) and (1.9), assume that  $(\rho, u, H)$  is a smooth solution of (1.1)-(1.4) on  $\mathbb{R}^2 \times (0, T]$  satisfying (3.4). Then, we derive some necessary uniform estimates on the spatial gradient of the smooth solution  $(\rho, u, H)$ .

**Lemma 4.1** *There is a positive constant  $C$  depending only on  $T, \mu, \lambda, \nu, \gamma, a, \bar{\rho}, \beta, N_0, M, q$ , and  $\|\rho_0\|_{H^1 \cap W^{1,q}}$  such that*

$$\begin{aligned}
&\sup_{0 \leq t \leq T} \left( \|\rho\|_{H^1 \cap W^{1,q}} + \|\nabla u\|_{L^2} + \|\nabla H\|_{L^2} + t \|\nabla^2 u\|_{L^2}^2 \right) \\
&\quad + \int_0^T \left( \|\nabla^2 u\|_{L^2}^2 + \|\nabla^2 H\|_{L^2}^2 + \|\nabla^2 u\|_{L^q}^{(q+1)/q} + t \|\nabla^2 u\|_{L^q}^2 \right) dt \leq C.
\end{aligned} \tag{4.1}$$

*Proof.* First, it follows from (3.53), (3.52), Gronwall's inequality, and (3.6) that

$$\sup_{t \in [0, T]} \left( \|\nabla u\|_{L^2}^2 + \|\nabla H\|_{L^2}^2 \right) + \int_0^T \left( \|\rho^{1/2} \dot{u}\|_{L^2}^2 + \|\Delta H\|_{L^2}^2 + \|H\| \|\nabla H\|_{L^2}^2 \right) dt \leq C, \tag{4.2}$$

which together with (2.9) and (3.7) shows

$$\int_0^T \left( \|\nabla u\|_{L^4}^4 + \|\nabla H\|_{L^4}^4 \right) dt \leq C. \tag{4.3}$$

Multiplying (3.33) by  $t$ , with the same arguments as (3.34)-(3.37), we have by (4.2) and (4.3) that

$$\sup_{0 \leq t \leq T} \left( t \|H\| \|\nabla H\|_{L^2}^2 \right) + \int_0^T \left( t \|\Delta H\|_{L^2}^2 + \|H\| \|\Delta H\|_{L^2}^2 \right) dt \leq C \tag{4.4}$$

Multiplying (3.28) by  $t$  and integrating the resulting inequality over  $(0, T)$  combined with (4.2), (4.3) and (4.4) lead to

$$\begin{aligned} & \sup_{0 \leq t \leq T} (t \|\rho^{1/2} \dot{u}\|_{L^2}^2 + t \|H\| \|\nabla H\|_{L^2}^2) \\ & + \int_0^T t (\|\nabla \dot{u}\|_{L^2}^2 + \|\Delta |H|^2\|_{L^2}^2 + \|H\| \|\Delta H\|_{L^2}^2) dt \leq C. \end{aligned} \quad (4.5)$$

Next, we prove (4.1) by using Lemma 2.7 as in [18]. For  $p \in [2, q]$ ,  $|\nabla \rho|^p$  satisfies

$$\begin{aligned} & (|\nabla \rho|^p)_t + \operatorname{div}(|\nabla \rho|^p u) + (p-1)|\nabla \rho|^p \operatorname{div} u \\ & + p|\nabla \rho|^{p-2} (\nabla \rho)^t \nabla u (\nabla \rho) + p\rho |\nabla \rho|^{p-2} \nabla \rho \cdot \nabla \operatorname{div} u = 0. \end{aligned}$$

Thus,

$$\begin{aligned} \frac{d}{dt} \|\nabla \rho\|_{L^p} & \leq C(1 + \|\nabla u\|_{L^\infty}) \|\nabla \rho\|_{L^p} + C \|\nabla^2 u\|_{L^p} \\ & \leq C(1 + \|\nabla u\|_{L^\infty}) \|\nabla \rho\|_{L^p} + C \|\rho \dot{u}\|_{L^p} + C \|H\| \|\nabla H\|_{L^p}, \end{aligned} \quad (4.6)$$

due to

$$\|\nabla^2 u\|_{L^p} \leq C(\|\rho \dot{u}\|_{L^p} + \|\nabla P\|_{L^p} + \|H\| \|\nabla H\|_{L^p}), \quad (4.7)$$

which follows from the standard  $L^p$ -estimate for the following elliptic system:

$$\mu \Delta u + (\mu + \lambda) \nabla \operatorname{div} u = \rho \dot{u} + \nabla P + \frac{1}{2} \nabla |H|^2 - \operatorname{div}(H \otimes H), \quad u \rightarrow 0 \text{ as } |x| \rightarrow \infty.$$

Next, it follows from the Gagliardo-Nirenberg inequality, (4.2), and (2.7) that

$$\begin{aligned} & \|\operatorname{div} u\|_{L^\infty} + \|\omega\|_{L^\infty} \\ & \leq C \|F\|_{L^\infty} + C \|P\|_{L^\infty} + C \|H\|^2_{L^\infty} + C \|\omega\|_{L^\infty} \\ & \leq C(q) + C(q) \|\nabla F\|_{L^q}^{q/(2(q-1))} + C(q) \|\nabla |H|^2\|_{L^q}^{q/(2(q-1))} + C(q) \|\nabla \omega\|_{L^q}^{q/(2(q-1))} \\ & \leq C(q) + C(q) (\|\rho \dot{u}\|_{L^q} + \|H\| \|\nabla H\|_{L^q})^{q/(2(q-1))}, \end{aligned} \quad (4.8)$$

which, together with Lemma 2.7, (4.7) and (4.2), yields that

$$\begin{aligned} \|\nabla u\|_{L^\infty} & \leq C(\|\operatorname{div} u\|_{L^\infty} + \|\omega\|_{L^\infty}) \log(e + \|\nabla^2 u\|_{L^q}) + C \|\nabla u\|_{L^2} + C \\ & \leq C \left( 1 + \|\rho \dot{u}\|_{L^q}^{q/(2(q-1))} + \|H\| \|\nabla H\|_{L^q}^{q/(2(q-1))} \right) \\ & \quad \cdot \log(e + \|\rho \dot{u}\|_{L^q} + \|H\| \|\nabla H\|_{L^q} + \|\nabla \rho\|_{L^q}) \\ & \leq C(1 + \|\rho \dot{u}\|_{L^q} + \|H\| \|\nabla H\|_{L^q}) \log(e + \|\nabla \rho\|_{L^q}). \end{aligned} \quad (4.9)$$

Next, it follows from the Hölder inequality and (3.120) that

$$\begin{aligned} \|\rho \dot{u}\|_{L^q} & \leq \|\rho \dot{u}\|_{L^2}^{2(q-1)/(q^2-2)} \|\rho \dot{u}\|_{L^{q^2}}^{q(q-2)/(q^2-2)} \\ & \leq C \|\rho \dot{u}\|_{L^2}^{2(q-1)/(q^2-2)} \left( \|\rho^{1/2} \dot{u}\|_{L^2} + \|\nabla \dot{u}\|_{L^2} \right)^{q(q-2)/(q^2-2)} \\ & \leq C \|\rho^{1/2} \dot{u}\|_{L^2} + C \|\rho^{1/2} \dot{u}\|_{L^2}^{2(q-1)/(q^2-2)} \|\nabla \dot{u}\|_{L^2}^{q(q-2)/(q^2-2)}, \end{aligned}$$

which combined with (4.2) and (4.5) implies that

$$\begin{aligned}
& \int_0^T \left( \|\rho \dot{u}\|_{L^q}^{1+1/q} + t \|\rho \dot{u}\|_{L^q}^2 \right) dt \\
& \leq C \int_0^T \left( \|\rho^{1/2} \dot{u}\|_{L^2}^2 + t \|\nabla \dot{u}\|_{L^2}^2 + t^{-(q^3-q^2-2q-1)/(q^3-q^2-2q)} \right) dt + C \\
& \leq C.
\end{aligned} \tag{4.10}$$

Moreover, we have by (3.119), (3.6) and (4.2) that

$$\begin{aligned}
& \int_0^T \left( \|H\|_{L^q} \|\nabla H\|_{L^q}^{1+\frac{1}{q}} + t \|H\|_{L^q} \|\nabla H\|_{L^q}^2 \right) dt \\
& \leq C \int_0^T \left( \|\nabla^2 H\|_{L^2}^{1-\frac{1}{q^2}} + t \|\nabla^2 H\|_{L^2}^{2-\frac{2}{q}} \right) dt \\
& \leq C \int_0^T (1 + t^q + \|\nabla^2 H\|_{L^2}^2) dt \leq C
\end{aligned} \tag{4.11}$$

Then, substituting (4.9) into (4.6) where  $p = q$ , we deduce from Gronwall's inequality, (4.2) and (4.10)-(4.11) that

$$\sup_{0 \leq t \leq T} \|\nabla \rho\|_{L^q} \leq C, \tag{4.12}$$

which, along with (4.7), (4.10) and (4.11), shows

$$\int_0^T \left( \|\nabla^2 u\|_{L^q}^{(q+1)/q} + t \|\nabla^2 u\|_{L^q}^2 \right) dt \leq C. \tag{4.13}$$

Finally, taking  $p = 2$  in (4.6), one gets by using (4.2), (4.13), and Gronwall's inequality that

$$\sup_{0 \leq t \leq T} \|\nabla \rho\|_{L^2} \leq C,$$

which, together with (4.12), (4.2), (4.7), (4.5), and (4.13), yields (4.1). The proof of Lemma 4.1 is completed.

**Lemma 4.2** *There is a positive constant  $C$  depending only on  $T, \mu, \lambda, \nu, \gamma, a, \bar{\rho}, \beta, N_0, M, q$ , and  $\|\nabla(\bar{x}^a \rho_0)\|_{L^2 \cap L^q}$  such that*

$$\sup_{0 \leq t \leq T} \|\bar{x}^a \rho\|_{L^1 \cap H^1 \cap W^{1,q}} \leq C. \tag{4.14}$$

*Proof.* First, it follows from (2.3), (3.121) and (3.63) that for any  $\eta \in (0, 1]$  and any  $s > 2$ ,

$$\|u \bar{x}^{-\eta}\|_{L^{s/\eta}} \leq C(\eta, s). \tag{4.15}$$

Multiplying (1.1)<sub>1</sub> by  $\bar{x}^a$  and integrating the resulting equality over  $\mathbb{R}^2$  lead to

$$\begin{aligned}
\frac{d}{dt} \int \rho \bar{x}^a dx & \leq C \int \rho |u| \bar{x}^{a-1} \log^2(e + |x|^2) dx \\
& \leq C \|\rho \bar{x}^{a-1+8/(8+a)}\|_{L^{(8+a)/(\gamma+a)}} \|u \bar{x}^{-4/(8+a)}\|_{L^{8+a}} \\
& \leq C \int \rho \bar{x}^a dx + C.
\end{aligned}$$

This gives

$$\sup_{0 \leq t \leq T} \int \rho \bar{x}^a dx \leq C. \quad (4.16)$$

Then, one derives from (1.1)<sub>1</sub> that  $v \triangleq \rho \bar{x}^a$  satisfies

$$v_t + u \cdot \nabla v - avu \cdot \nabla \log \bar{x} + v \operatorname{div} u = 0,$$

which, together with some estimates as for (4.6), gives that for any  $p \in [2, q]$

$$\begin{aligned} (\|\nabla v\|_{L^p})_t &\leq C(1 + \|\nabla u\|_{L^\infty} + \|u \cdot \nabla \log \bar{x}\|_{L^\infty}) \|\nabla v\|_{L^p} \\ &\quad + C\|v\|_{L^\infty} (\|\nabla u\|_{L^p} \|\nabla \log \bar{x}\|_{L^p} + \|u\|_{L^p} \|\nabla^2 \log \bar{x}\|_{L^p} + \|\nabla^2 u\|_{L^p}) \\ &\leq C(1 + \|\nabla u\|_{W^{1,q}}) \|\nabla v\|_{L^p} \\ &\quad + C\|v\|_{L^\infty} \left( \|\nabla u\|_{L^p} + \|u \bar{x}^{-2/5}\|_{L^{4p}} \|\bar{x}^{-3/2}\|_{L^{4p/3}} + \|\nabla^2 u\|_{L^p} \right) \\ &\leq C(1 + \|\nabla^2 u\|_{L^p} + \|\nabla u\|_{W^{1,q}}) (1 + \|\nabla v\|_{L^p} + \|\nabla v\|_{L^q}), \end{aligned} \quad (4.17)$$

where in the second and the last inequalities, one has used (4.15) and (4.16). Choosing  $p = q$  in (4.17), we obtain after using Gronwall's inequality and (4.1) that

$$\sup_{0 \leq t \leq T} \|\nabla(\rho \bar{x}^a)\|_{L^q} \leq C. \quad (4.18)$$

Finally, setting  $p = 2$  in (4.17), we deduce from (4.1) and (4.18) that

$$\sup_{0 \leq t \leq T} \|\nabla(\rho \bar{x}^a)\|_{L^2} \leq C,$$

which combined with (4.16) and (4.18) thus gives (4.14) and finishes the proof of Lemma 4.2.

**Lemma 4.3** *There exists a positive constant  $C$  depending only on  $T, \mu, \lambda, \nu, \gamma, a, \bar{\rho}, \beta, N_0, M, q$ , and  $\|H_0\|^2 \bar{x}^a\|_{L^1}$  such that*

$$\sup_{0 \leq t \leq T} \|H \bar{x}^{\frac{a}{2}}\|_{L^2}^2 + \int_0^T \|\nabla H \bar{x}^{\frac{a}{2}}\|_{L^2}^2 dt \leq C, \quad (4.19)$$

$$\sup_{0 \leq t \leq T} \left( t \|\nabla H \bar{x}^{\frac{a}{2}}\|_{L^2}^2 \right) + \int_0^T t \|\Delta H \bar{x}^{\frac{a}{2}}\|_{L^2}^2 dt \leq C. \quad (4.20)$$

*Proof.* In order to prove (4.19), we will following the same arguments as (3.74) (or [31, Lemma 4.1]). Multiplying (1.1)<sub>3</sub> by  $H \bar{x}^a$ , integrating the resulting equation by parts over  $\mathbb{R}^2$ , together with (4.2) and (4.15), it holds that

$$\frac{1}{2} \left( \|H \bar{x}^{\frac{a}{2}}\|_{L^2}^2 \right)_t + \nu \|\nabla H \bar{x}^{\frac{a}{2}}\|_{L^2}^2 \leq C \|H \bar{x}^{\frac{a}{2}}\|_{L^2}^2 + \varepsilon \|\nabla H \bar{x}^{\frac{a}{2}}\|_{L^2}^2, \quad (4.21)$$

then choosing  $\varepsilon$  suitably small, together with Gronwall's inequality yields (4.19).

Now, multiplying (1.1)<sub>3</sub> by  $\Delta H \bar{x}^a$ , integrating the resultant equation by parts over  $\mathbb{R}^2$ , it follows from the similar arguments as (3.19) that

$$\begin{aligned} &\frac{1}{2} \left( \int |\nabla H|^2 \bar{x}^a dx \right)_t + \nu \int |\Delta H|^2 \bar{x}^a dx \\ &\leq C \int |\nabla H| |H| |\nabla u| |\nabla \bar{x}^a| dx + C \int |\nabla H|^2 |u| |\nabla \bar{x}^a| dx + C \int |\nabla H| |\Delta H| \bar{x}^a dx \\ &\quad + C \int |H| |\nabla u| |\Delta H| \bar{x}^a dx + C \int |\nabla u| |\nabla H|^2 \bar{x}^a dx \triangleq \sum_{i=1}^5 \tilde{J}_i. \end{aligned} \quad (4.22)$$



The five terms on the right of (4.22) can be estimated as follows:

$$\begin{aligned}
\tilde{J}_1 &\leq C \int |\nabla H| |H| |\nabla u| \bar{x}^a (\bar{x}^{-1} |\nabla \bar{x}|) dx \\
&\leq C \|H \bar{x}^{a/2}\|_{L^4}^4 + C \|\nabla u\|_{L^4}^4 + C \|\nabla H \bar{x}^{a/2}\|_{L^2}^2 \\
&\leq C \|H \bar{x}^{a/2}\|_{L^2}^2 \left( \|\nabla H \bar{x}^{a/2}\|_{L^2}^2 + \|H \bar{x}^{a/2}\|_{L^2}^2 \right) + C \|\nabla u\|_{L^4}^4 + C \|\nabla H \bar{x}^{a/2}\|_{L^2}^2 \\
&\leq C + C \|\nabla^2 u\|_{L^2}^2 + C \|\nabla H \bar{x}^{a/2}\|_{L^2}^2
\end{aligned} \tag{4.23}$$

owing to (4.19) and (4.1),

$$\begin{aligned}
\tilde{J}_2 &\leq C \int |\nabla H|^{2-\frac{1}{a}} \bar{x}^{a-\frac{1}{2}} |\nabla H|^{\frac{1}{a}} |u| \bar{x}^{-\frac{1}{4}} \bar{x}^{-\frac{1}{4}} |\nabla \bar{x}| dx \\
&\leq C \| |\nabla H|^{2-\frac{1}{a}} \bar{x}^{a-\frac{1}{2}} \|_{L^{\frac{2a}{2a-1}}} \|u \bar{x}^{-\frac{1}{4}}\|_{L^{4a}} \| |\nabla H|^{\frac{1}{a}} \|_{L^{4a}} \\
&\leq C \|\nabla H \bar{x}^{\frac{a}{2}}\|_{L^2}^2 + C \|\nabla H\|_{L^4}^2 \\
&\leq C \|\nabla H \bar{x}^{\frac{a}{2}}\|_{L^2}^2 + \varepsilon \|\Delta H \bar{x}^{\frac{a}{2}}\|_{L^2}^2
\end{aligned} \tag{4.24}$$

owing to (4.15) and the fact  $|\bar{x}| > 1$ ,

$$\begin{aligned}
\tilde{J}_3 + \tilde{J}_4 &\leq \varepsilon \|\Delta H \bar{x}^{a/2}\|_{L^2}^2 + C \|\nabla H \bar{x}^{a/2}\|_{L^2}^2 + C \|H \bar{x}^{a/2}\|_{L^4}^4 + C \|\nabla u\|_{L^4}^4 \\
&\leq \varepsilon \|\Delta H \bar{x}^{a/2}\|_{L^2}^2 + C + C \|\nabla H \bar{x}^{a/2}\|_{L^2}^2 + C \|\nabla^2 u\|_{L^2}^2
\end{aligned} \tag{4.25}$$

owing to (4.19) and (4.1),

$$\begin{aligned}
\tilde{J}_5 &\leq C \|\nabla u\|_{L^\infty} \|\nabla H \bar{x}^{a/2}\|_{L^2}^2 \leq C \|\nabla u\|_{L^2}^{(r-2)/(2r-2)} \|\nabla^2 u\|_{L^r}^{r/(2r-2)} \|\nabla H \bar{x}^{a/2}\|_{L^2}^2 \\
&\leq C (1 + \|\nabla^2 u\|_{L^r}^{(r+1)/r}) \|\nabla H \bar{x}^{a/2}\|_{L^2}^2
\end{aligned} \tag{4.26}$$

owing to (2.2) and (4.1). Submitting (4.23)-(4.26) into (4.22) and choosing  $\varepsilon$  suitably small, we have

$$\begin{aligned}
&\frac{1}{2} \left( \int |\nabla H|^2 \bar{x}^a dx \right)_t + \nu \int |\Delta H|^2 \bar{x}^a dx \\
&\leq C (1 + \|\nabla^2 u\|_{L^r}^{(r+1)/r}) \|\nabla H \bar{x}^{a/2}\|_{L^2}^2 + C (\|\nabla^2 u\|_{L^2}^2 + 1),
\end{aligned} \tag{4.27}$$

which multiplied by  $t$ , then together with Gronwall's inequality, (4.19) and (4.1) yields (4.20). The proof of Lemma 4.3 is finished.

**Lemma 4.4** *There is a positive constant  $C$  depending only on  $T, \mu, \lambda, \nu, \gamma, a, \bar{\rho}, \beta, N_0, M, q$ ,  $\|H_0\|^2 \bar{x}^a\|_{L^1}$  and  $\|\nabla(\bar{x}^a \rho_0)\|_{L^2 \cap L^q}$  such that*

$$\sup_{0 \leq t \leq T} t \left( \|\rho^{1/2} u_t\|_{L^2}^2 + \|H_t\|_{L^2}^2 \right) + \int_0^T t \|\nabla u_t\|_{L^2}^2 + t \|\nabla H_t\|_{L^2}^2 dt \leq C, \tag{4.28}$$

$$\sup_{0 \leq t \leq T} t (\|\nabla u\|_{H^1}^2 + \|\nabla H\|_{H^1}^2) \leq C. \tag{4.29}$$

*Proof.* First, the combination of (4.15) with (4.14) gives that for any  $\eta \in (0, 1]$  and any  $s > 2$ ,

$$\|\rho^\eta u\|_{L^{s/\eta}} + \|u \bar{x}^{-\eta}\|_{L^{s/\eta}} \leq C(\eta, s). \tag{4.30}$$

Multiplying equations (1.1)<sub>2</sub> by  $u_t$  and integration by parts yield

$$\begin{aligned} & \frac{d}{dt} \int ((\mu + \lambda)(\operatorname{div} u)^2 + \mu|\nabla u|^2) dx + \int \rho|u_t|^2 dx \\ & \leq 2 \int P \operatorname{div} u_t dx + C \int \rho|u|^2 |\nabla u|^2 dx + \int \left( H \cdot \nabla H - \frac{1}{2} |\nabla H|^2 \right) u_t dx, \end{aligned} \quad (4.31)$$

where

$$\int \rho|u|^2 |\nabla u|^2 dx \leq C \|\rho^{1/2} u\|_{L^4}^2 \|\nabla u\|_{L^4}^2 \leq C + \|\nabla^2 u\|_{L^2}^2 \quad (4.32)$$

due to (4.30) and (4.1),

$$\begin{aligned} 2 \int P \operatorname{div} u_t dx &= 2 \frac{d}{dt} \int P \operatorname{div} u dx - 2 \int P_t \operatorname{div} u dx \\ &= 2 \frac{d}{dt} \int P \operatorname{div} u dx - 2 \int P u \cdot \nabla \operatorname{div} u dx + 2(\gamma - 1) \int P (\operatorname{div} u)^2 dx \\ &\leq 2 \frac{d}{dt} \int P \operatorname{div} u dx + C \|\rho^\gamma u\|_{L^2}^2 + C \|\nabla^2 u\|_{L^2}^2 + C \|\nabla u\|_{L^2}^2 \\ &\leq 2 \frac{d}{dt} \int P \operatorname{div} u dx + C \|\nabla^2 u\|_{L^2}^2 + C. \end{aligned} \quad (4.33)$$

due to (3.13), (4.30), (4.1), and (4.14),

$$\begin{aligned} & \int \left( H \cdot \nabla H - \frac{1}{2} |\nabla H|^2 \right) u_t dx = - \int H \cdot \nabla u_t \cdot H - \frac{1}{2} H^2 \operatorname{div} u_t dx \\ & \leq \frac{d}{dt} \left( \int \frac{1}{2} |H|^2 \operatorname{div} u dx - \int H \cdot \nabla u \cdot H dx \right) + \int |H_t| |H| |\nabla u| dx \\ & \leq \frac{d}{dt} \left( \int \frac{1}{2} |H|^2 \operatorname{div} u dx - \int H \cdot \nabla u \cdot H dx \right) + \varepsilon \|H_t\|_{L^2}^2 + C \|H\|_{L^4}^2 \|\nabla u\|_{L^4}^2 \\ & \leq \frac{d}{dt} \left( \int \frac{1}{2} |H|^2 \operatorname{div} u dx - \int H \cdot \nabla u \cdot H dx \right) + \varepsilon \|H_t\|_{L^2}^2 + C \|\nabla^2 u\|_{L^2}^2 + C \end{aligned} \quad (4.34)$$

owing to (4.1). Putting (4.32), (4.34), and (4.33) into (4.31) gives

$$\begin{aligned} & \frac{d}{dt} \int ((\mu + \lambda)(\operatorname{div} u)^2 + \mu|\nabla u|^2) dx + \int \rho|u_t|^2 dx \\ & \leq \varepsilon \|H_t\|_{L^2}^2 + C \|\nabla^2 u\|_{L^2}^2 + C + \frac{d}{dt} \bar{B}(t), \end{aligned} \quad (4.35)$$

where

$$\begin{aligned} \bar{B}(t) &= \int P \operatorname{div} u dx + \int \frac{1}{2} |H|^2 \operatorname{div} u dx - \int H \cdot \nabla u \cdot H dx \\ &\leq C \|H\|_{L^4}^4 + C \|P\|_{L^2}^2 + C \|\nabla u\|_{L^2}^2 \leq C, \end{aligned} \quad (4.36)$$

due to (4.1) and (4.14). Moreover, it follows from (1.1)<sub>3</sub> that

$$\begin{aligned} \frac{d}{dt} \|\nabla H\|_{L^2}^2 + \|H_t\|_{L^2}^2 + \|\Delta H\|_{L^2}^2 &\leq C \|H\| \|\nabla u\|_{L^2}^2 + C \|u\| \|\nabla H\|_{L^2}^2 \\ &\leq C \|H\|_{L^4}^2 \|\nabla u\|_{L^4}^2 + C \|u\| \|\nabla H\|_{L^2}^2 \\ &\leq C + \|\nabla^2 u\|_{L^2}^2 + C \|u\| \|\nabla H\|_{L^2}^2 \end{aligned} \quad (4.37)$$

which together with (4.35) and choosing  $\varepsilon$  suitably small yield that

$$\begin{aligned} & \frac{d}{dt} (\|\nabla u\|_{L^2}^2 + \|\nabla H\|_{L^2}^2) + \|\rho^{\frac{1}{2}} u_t\|_{L^2}^2 + \|H_t\|_{L^2}^2 + \|\Delta H\|_{L^2}^2 \\ & \leq C + \|\nabla^2 u\|_{L^2}^2 + C\|u\|\|\nabla H\|_{L^2}^2 + \frac{d}{dt} \bar{B}(t) \end{aligned} \quad (4.38)$$

By (4.36), (4.1) and (4.14), integrating (4.38) over  $(0, T)$  yields that

$$\sup_{0 \leq t \leq T} (\|\nabla u\|_{L^2}^2 + \|\nabla H\|_{L^2}^2) + \int_0^T (\|\rho^{\frac{1}{2}} u_t\|_{L^2}^2 + \|H_t\|_{L^2}^2 + \|\Delta H\|_{L^2}^2) dt \leq C, \quad (4.39)$$

where one has used following estimate

$$\int_0^T \|u\|\|\nabla H\|_{L^2}^2 dt \leq C \quad (4.40)$$

which is deduced directly from (4.2) and (4.19), i.e.,

$$\begin{aligned} \|u\|\|\nabla H\|_{L^2}^2 &= \int |u|^2 \bar{x}^{-1/2} |\nabla H| \bar{x}^{1/2} |\nabla H| dx \\ &\leq C \|u \bar{x}^{-\frac{1}{4}}\|_{L^8}^4 \|\nabla H\|_{L^4}^2 + C \|\nabla H \bar{x}^{\frac{1}{2}}\|_{L^2}^2 \\ &\leq C \|\nabla H\|_{L^2}^2 + \frac{1}{2} \|\nabla^2 H\|_{L^2}^2 + C \|\nabla H \bar{x}^{\frac{\alpha}{2}}\|_{L^2}^2 \end{aligned} \quad (4.41)$$

owing to (4.15).

Now, differentiating (1.1)<sub>2</sub> with respect to  $t$ , and multiplying the resulting equation by  $u_t$ , then integrating over  $\mathbb{R}^2$ , we obtain after using (1.1)<sub>1</sub> that

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int \rho |u_t|^2 dx + \int (\mu |\nabla u_t|^2 + (\mu + \lambda) (\operatorname{div} u_t)^2) dx \\ &= -2 \int \rho u \cdot \nabla u_t \cdot u_t dx - \int \rho u \cdot \nabla (u \cdot \nabla u \cdot u_t) dx \\ & \quad - \int \rho u_t \cdot \nabla u \cdot u_t dx + \int P_t \operatorname{div} u_t dx + \int \left( H \cdot \nabla H - \frac{1}{2} \nabla |H|^2 \right)_t u_t dx \\ &\leq C \int \rho |u| |u_t| (|\nabla u_t| + |\nabla u|^2 + |u| |\nabla^2 u|) dx + C \int \rho |u|^2 |\nabla u| |\nabla u_t| dx \\ & \quad + C \int \rho |u_t|^2 |\nabla u| dx + C(\delta) \|P_t\|_{L^2}^2 + \delta \|\nabla u_t\|_{L^2}^2 + \int \left( H \cdot \nabla H - \frac{1}{2} \nabla |H|^2 \right)_t u_t dx. \end{aligned} \quad (4.42)$$

Each term on the right-hand side of (4.42) can be estimated as follows:

Moreover, it follows from (2.5), (3.121), and (3.63) that

$$\|\rho^{1/2} u_t\|_{L^6} \leq C \|\rho^{1/2} u_t\|_{L^2} + C \|\nabla u_t\|_{L^2}, \quad (4.43)$$

which together with (4.30), (4.2), and Holder's inequality yields that for  $\delta \in (0, 1)$ ,

$$\begin{aligned} & \int \rho |u| |u_t| (|\nabla u_t| + |\nabla u|^2 + |u| |\nabla^2 u|) dx \\ & \leq C \|\rho^{1/2} u\|_{L^6} \|\rho^{1/2} u_t\|_{L^2}^{1/2} \|\rho^{1/2} u_t\|_{L^6}^{1/2} (\|\nabla u_t\|_{L^2} + \|\nabla u\|_{L^4}^2) \\ & \quad + C \|\rho^{1/4} u\|_{L^{12}}^2 \|\rho^{1/2} u_t\|_{L^2}^{1/2} \|\rho^{1/2} u_t\|_{L^6}^{1/2} \|\nabla^2 u\|_{L^2} \\ & \leq C \|\rho^{1/2} u_t\|_{L^2}^{1/2} \left( \|\rho^{1/2} u_t\|_{L^2} + \|\nabla u_t\|_{L^2} \right)^{1/2} (\|\nabla u_t\|_{L^2} + \|\nabla^2 u\|_{L^2} + 1) \\ & \leq \delta \|\nabla u_t\|_{L^2}^2 + C(\delta) \left( \|\nabla^2 u\|_{L^2}^2 + \|\rho^{1/2} u_t\|_{L^2}^2 + 1 \right). \end{aligned} \quad (4.44)$$

Next, Holder's inequality, (4.30), and (4.43) lead to

$$\begin{aligned}
& \int \rho |u|^2 |\nabla u| |\nabla u_t| dx + \int \rho |u_t|^2 |\nabla u| dx \\
& \leq C \|\rho^{1/2} u\|_{L^8}^2 \|\nabla u\|_{L^4} \|\nabla u_t\|_{L^2} + \|\nabla u\|_{L^2} \|\rho^{1/2} u_t\|_{L^6}^{3/2} \|\rho^{1/2} u_t\|_{L^2}^{1/2} \\
& \leq \delta \|\nabla u_t\|_{L^2}^2 + C(\delta) \left( \|\nabla^2 u\|_{L^2}^2 + \|\rho^{1/2} u_t\|_{L^2}^2 + 1 \right).
\end{aligned} \tag{4.45}$$

It follows from (3.13), (4.30), (4.2), and (4.14) that

$$\begin{aligned}
\|P_t\|_{L^2} & \leq C \|\bar{x}^{-1/2} u\|_{L^{2q/(q-2)}} \|\rho\|_{L^\infty}^{\gamma-1} \|\bar{x}^{1/2} \nabla \rho\|_{L^q} + C \|\nabla u\|_{L^2} \\
& \leq C \|\bar{x}^a \nabla \rho\|_{L^q} + C \leq C.
\end{aligned} \tag{4.46}$$

Moreover, we obtain after integration by parts that

$$\begin{aligned}
& \int \left( H \cdot \nabla H - \frac{1}{2} \nabla |H|^2 \right)_t u_t dx \\
& = - \int H_t \cdot \nabla u_t \cdot H dx - \int H \cdot \nabla u_t \cdot H_t dx + \int H \cdot H_t \operatorname{div} u_t dx \triangleq S_1.
\end{aligned} \tag{4.47}$$

Finally, putting (4.44)-(4.47) into (4.42) and choosing  $\delta$  suitably small, we obtain after using (4.7) and (4.1) that

$$\begin{aligned}
& \frac{d}{dt} \int \rho |u_t|^2 dx + \mu \int |\nabla u_t|^2 dx \\
& \leq C \int \rho |u_t|^2 dx + C \int \rho |\dot{u}|^2 dx + C \int |H|^2 |\nabla H|^2 dx + C + S_1
\end{aligned} \tag{4.48}$$

Next, differentiating (1.1)<sub>3</sub> with respect to  $t$  shows

$$H_{tt} - H_t \cdot \nabla u - H \cdot \nabla u_t + u_t \cdot \nabla H + u \cdot \nabla H_t + H_t \operatorname{div} u + H \operatorname{div} u_t = \nu \Delta H_t. \tag{4.49}$$

Multiplying (4.49) by  $H_t$  and integrating the resulting equation over  $\mathbb{R}^2$ , yields that

$$\begin{aligned}
\frac{1}{2} \frac{d}{dt} \int |H_t|^2 dx + \nu \int |\nabla H_t|^2 dx & = \int (H \cdot \nabla u_t - u_t \cdot \nabla H - H \operatorname{div} u_t) \cdot H_t dx \\
& + \int (H_t \cdot \nabla u - u \cdot \nabla H_t - H_t \operatorname{div} u) \cdot H_t dx \triangleq S_2 + S_3,
\end{aligned} \tag{4.50}$$

where

$$\begin{aligned}
S_2 & = \int H \cdot \nabla u_t \cdot H_t dx - \int u_t \cdot \nabla H \cdot H_t - \int H \cdot H_t \operatorname{div} u_t dx, \\
S_3 & = \int H_t \cdot \nabla u \cdot H_t dx - \int u \cdot \nabla H_t \cdot H_t dx - \int |H_t|^2 \operatorname{div} u dx \\
& \leq C \int |H_t|^2 |\nabla u| dx \leq C \|H_t\|_{L^4}^2 \|\nabla u\|_{L^2} \leq C \|H_t\|_{L^2} \|\nabla H_t\|_{L^2} \|\nabla u\|_{L^2} \\
& \leq \varepsilon \|\nabla H_t\|_{L^2}^2 + C \|H_t\|_{L^2}^2,
\end{aligned} \tag{4.51}$$

due to (4.1), which together with (4.50), and choosing  $\varepsilon$  suitably small lead to

$$\frac{1}{2} \frac{d}{dt} \int |H_t|^2 dx + \nu \int |\nabla H_t|^2 dx \leq C \|H_t\|_{L^2}^2 + S_2 \tag{4.52}$$

Recall the notion  $S_1$  and  $S_2$  in (4.47) and (4.50) respectively, adding (4.48) and (4.52) together, and multiplying the resultant inequality by  $t$ , it holds by (4.5) that

$$\begin{aligned} & \frac{d}{dt} \left( t \|\rho^{\frac{1}{2}} u_t\|_{L^2}^2 + t \|H_t\|_{L^2}^2 \right) + t \|\nabla u_t\|_{L^2}^2 + t \|\nabla H_t\|_{L^2}^2 \\ & \leq C \left( t \|\rho^{\frac{1}{2}} u_t\|_{L^2}^2 + t \|H_t\|_{L^2}^2 \right) + \left( \|\rho^{\frac{1}{2}} u_t\|_{L^2}^2 + \|H_t\|_{L^2}^2 \right) + C \\ & \quad - t \int H_t \cdot \nabla u_t \cdot H dx - t \int u_t \cdot \nabla H \cdot H_t dx. \end{aligned} \quad (4.53)$$

The last two terms on the right-hand side of (4.53) can be estimated as follows:

$$\begin{aligned} -t \int H_t \cdot \nabla u_t \cdot H dx & \leq t \|H_t\|_{L^4} \|H\|_{L^4} \|\nabla u_t\|_{L^2} \\ & \leq \varepsilon t \|\nabla u_t\|_{L^2}^2 + C(\varepsilon) t \|H_t\|_{L^2} \|\nabla H_t\|_{L^2} \|H\|_{L^4}^2 \\ & \leq \varepsilon t \|\nabla u_t\|_{L^2}^2 + \varepsilon t \|\nabla H_t\|_{L^2}^2 + C t \|H_t\|_{L^2}^2, \end{aligned} \quad (4.54)$$

owing to Young's inequality, (2.1) and (3.7). Note  $\dot{u} = u_t + u \cdot \nabla u$ , it holds

$$\begin{aligned} -t \int u_t \cdot \nabla H \cdot H_t dx & = t \int \dot{u} \cdot \nabla H_t \cdot H dx + t \int H \cdot \nabla \dot{u} \cdot H_t dx \\ & \quad + t \int u \cdot \nabla u \cdot \nabla H \cdot H_t dx \triangleq l_1 + l_2 + l_3, \end{aligned} \quad (4.55)$$

and by (2.5), (3.7), (4.19), (2.1), (3.46) and (4.1), it holds

$$\begin{aligned} l_1 & \leq C t \int |\dot{u}|^2 |H|^2 dx + \varepsilon t \|\nabla H_t\|_{L^2}^2 \\ & \leq C t \int |\dot{u}|^2 \bar{x}^{-1/2} |H| \bar{x}^{1/2} |H| dx + \varepsilon t \|\nabla H_t\|_{L^2}^2 \\ & \leq C t \|\dot{u}\|_{L^4}^2 \|\bar{x}^{-1/2}\|_{L^4} \|H \bar{x}^{1/2}\|_{L^2} \|H\|_{L^4} + \varepsilon t \|\nabla H_t\|_{L^2}^2 \\ & \leq C t \|\dot{u} \bar{x}^{-1/4}\|_{L^8}^2 \|H \bar{x}^{1/2}\|_{L^2} \|H\|_{L^4} + \varepsilon t \|\nabla H_t\|_{L^2}^2 \\ & \leq C t \left( \|\rho^{\frac{1}{2}} \dot{u}\|_{L^2}^2 + \|\nabla \dot{u}\|_{L^2}^2 \right) + \varepsilon t \|\nabla H_t\|_{L^2}^2 \\ l_2 & \leq t \|H\|_{L^4} \|H_t\|_{L^4} \|\nabla \dot{u}\|_{L^2} \leq \varepsilon t \|\nabla H_t\|_{L^2}^2 + C t \|H_t\|_{L^2}^2 + C t \|\nabla \dot{u}\|_{L^2}^2 \\ J_3 & \leq \|u\| \|\nabla H\|_{L^2}^2 + C t^2 \|\nabla u\|_{L^4}^2 \|H_t\|_{L^4}^2 \\ & \leq \|u\| \|\nabla H\|_{L^2}^2 + C t^2 \|\nabla u\|_{L^2} \|\nabla^2 u\|_{L^2} \|H_t\|_{L^2} \|\nabla H_t\|_{L^2} \\ & \leq \|u\| \|\nabla H\|_{L^2}^2 + C(\varepsilon) t \|\nabla u\|_{L^2}^2 t \|\nabla^2 u\|_{L^2}^2 t \|H_t\|_{L^2}^2 + \varepsilon t \|\nabla H_t\|_{L^2}^2 \\ & \leq \|u\| \|\nabla H\|_{L^2}^2 + C(t \|H_t\|_{L^2}^2) + \varepsilon t \|\nabla H_t\|_{L^2}^2. \end{aligned} \quad (4.56)$$

Now, Putting (4.54) into (4.53) and choosing  $\varepsilon$  suitably small, then we can obtain (4.28) after using Gronwall's inequality, (4.1), (4.5), (4.39) and (4.40).

Finally, notice that

$$\begin{aligned} \|\nabla u\|_{H^1}^2 + \|\nabla H\|_{H^1}^2 & \leq C \|\nabla u\|_{L^2}^2 + C \|\nabla H\|_{L^2}^2 + C \|\nabla^2 u\|_{L^2}^2 + C \|\nabla^2 H\|_{L^2}^2 \\ & \leq C \|\nabla u\|_{L^2}^2 + C \|\nabla H\|_{L^2}^2 + C \|\nabla^2 u\|_{L^2}^2 + C \|H_t\|_{L^2}^2 \\ & \quad + C \|u\| \|\nabla H\|_{L^2}^2 + C \|H\| \|\nabla u\|_{L^2}^2 \\ & \leq C \|\nabla u\|_{L^2}^2 + C \|\nabla H\|_{L^2}^2 + C \|\nabla^2 u\|_{L^2}^2 \\ & \quad + C \|H_t\|_{L^2}^2 + \frac{1}{2} \|\nabla^2 H\|_{L^2}^2 + C \|\nabla H \bar{x}^{\frac{a}{2}}\|_{L^2}^2 \end{aligned} \quad (4.57)$$

where in the second and last inequalities one has used respectively (1.1)<sub>3</sub> and (4.41). Multiplying (4.57) by  $t$ , we obtain (4.29) directly from (4.1), (4.28) and (4.20).

The proof of Lemma 4.4 is finished.

## 5 Proofs of Theorems 1.1 and 1.2

With all the a priori estimates in Sections 3 and 4 at hand, we are ready to prove the main result of this paper in this section.

*Proof of Theorem 1.1.* By Lemma 2.1, there exists a  $T_* > 0$  such that the Cauchy problem (1.1)–(1.4) has a unique strong solution  $(\rho, u, H)$  on  $\mathbb{R}^2 \times (0, T_*]$ . We will use the a priori estimates, Proposition 3.1 and Lemmas 4.1–4.4, to prove the local strong solution  $(\rho, u, H)$  shall exist for all time.

First, it follows from (3.1), (3.2), (3.3), and (1.8) that

$$A_1(0) + A_2(0) = 0, \quad A_3(0) = 0, \quad \rho_0 \leq \bar{\rho}.$$

Therefore, there exists a  $T_1 \in (0, T_*]$  such that (3.4) holds for  $T = T_1$ .

Next, set

$$T^* = \sup\{T \mid (3.4) \text{ holds}\}. \quad (5.1)$$

Then  $T^* \geq T_1 > 0$ . Hence, for any  $0 < \tau < T \leq T^*$  with  $T$  finite, one deduces from (4.28) and (4.29) that

$$\nabla u, \nabla H \in C([\tau, T]; L^2 \cap L^q), \quad (5.2)$$

where one has used the standard embedding

$$L^\infty(\tau, T; H^1) \cap H^1(\tau, T; H^{-1}) \hookrightarrow C(\tau, T; L^q), \quad \text{for any } q \in [2, \infty].$$

Moreover, it follows from (4.1), (4.14) and [28, Lemma 2.3] that

$$\rho \in C([0, T]; L^1 \cap H^1 \cap W^{1,q}). \quad (5.3)$$

Finally, we claim that

$$T^* = \infty. \quad (5.4)$$

Otherwise,  $T^* < \infty$ . Then by Proposition 3.1, (3.5) holds for  $T = T^*$ . It follows from (3.6), (4.14), (4.19), (5.2) and (5.3) that  $(\rho(x, T^*), u(x, T^*), H(x, T^*))$  satisfies (1.8) except  $(u(\cdot, T^*), H(\cdot, T^*)) \in \dot{H}^\beta$ . Thus, Lemma 2.1 implies that there exists some  $T^{**} > T^*$ , such that (3.4) holds for  $T = T^{**}$ , which contradicts (5.1). Hence, (5.4) holds. Lemmas 2.1 and 4.1–4.4 thus show that  $(\rho, u, H)$  is in fact the unique strong solution defined on  $\mathbb{R}^2 \times (0, T]$  for any  $0 < T < T^* = \infty$ . Thus, the proof of Theorem 1.1 is completed.

To prove Theorem 1.2, we need the following elementary estimates similar to those of Lemma 2.5 whose proof can be found in [27, Lemma 2.2].

**Lemma 5.1** *Let  $\Omega = \mathbb{R}^3$  and  $(\rho, u, H)$  be a smooth solution of (1.1). Then there exists a generic positive constant  $C$  depending only on  $\mu, \lambda$  and  $\nu$  such that for any  $p \in [2, 6]$*

$$\|\nabla F\|_{L^p} + \|\nabla \omega\|_{L^p} \leq C (\|\rho \dot{u}\|_{L^p} + \|H\| \|\nabla H\|_{L^p}), \quad (5.5)$$

$$\begin{aligned} \|F\|_{L^p} + \|\omega\|_{L^p} &\leq C (\|\rho\dot{u}\|_{L^2} + \|H\|\|\nabla H\|_{L^2})^{(3p-6)/(2p)} \\ &\quad \cdot (\|\nabla u\|_{L^2} + \|P\|_{L^2} + \|H\|_{L^4}^2)^{(6-p)/(2p)}, \end{aligned} \quad (5.6)$$

$$\|\nabla u\|_{L^p} \leq C (\|F\|_{L^p} + \|\omega\|_{L^p}) + C\|P\|_{L^p} + C\|H\|^2_{L^p}, \quad (5.7)$$

where  $F$  and  $\omega$  are defined in (1.16).

Moreover, we state the following well-known Gagliardo-Nirenberg inequality (see [33]) in  $\mathbb{R}^3$ :

**Lemma 5.2 (Gagliardo-Nirenberg-3D)** *For  $p \in [2, 6]$ ,  $q \in (1, \infty)$ , and  $r \in (3, \infty)$ , there exists some generic constant  $C > 0$  which may depend on  $p, q$ , and  $r$  such that for  $f \in H^1(\mathbb{R}^3)$  and  $g \in L^q(\mathbb{R}^3) \cap D^{1,r}(\mathbb{R}^3)$ , we have*

$$\|f\|_{L^p(\mathbb{R}^3)}^p \leq C \|f\|_{L^2(\mathbb{R}^3)}^{\frac{6-p}{2}} \|\nabla f\|_{L^2(\mathbb{R}^3)}^{\frac{3p-6}{2}}, \quad (5.8)$$

$$\|g\|_{L^\infty(\mathbb{R}^3)} \leq C \|g\|_{L^q(\mathbb{R}^3)}^{\frac{q(r-3)}{3r+q(r-3)}} \|\nabla g\|_{L^r(\mathbb{R}^3)}^{\frac{3r}{3r+q(r-3)}}. \quad (5.9)$$

*Proof of Theorem 1.2.* It suffices to prove (1.24). In fact, it follows from [27, Proposition 3.1 and (3.10)] that there exists some  $\varepsilon$  depending only on  $\mu, \nu, \lambda, \gamma, \bar{\rho}, \beta$ , and  $M$  such that

$$\begin{aligned} &\sup_{1 \leq t < \infty} (\|H\|_{L^2} + \|H\|_{L^3} + \|\rho\|_{L^\gamma \cap L^\infty} \\ &\quad + \|\nabla u\|_{L^2} + \|\nabla H\|_{L^2} + \|\rho^{1/2}\dot{u}\|_{L^2} + \|\nabla^2 H\|_{L^2} + \|H_t\|_{L^2}) \\ &+ \int_1^\infty (\|\nabla u\|_{L^2}^2 + \|\nabla H\|_{L^2}^2 + \| |H|^{1/2} \nabla H \|_{L^2}^2 \\ &\quad + \|\rho^{1/2}\dot{u}\|_{L^2}^2 + \|\nabla^2 H\|_{L^2}^2 + \|H_t\|_{L^2}^2 + \|\nabla \dot{u}\|_{L^2}^2 + \|\nabla H_t\|_{L^2}^2) dt \leq C, \end{aligned} \quad (5.10)$$

provided  $C_0 \leq \varepsilon$ .

If  $1 < \gamma \leq 3/2$ , it then holds that

$$\sup_{1 \leq t < \infty} \|\rho\|_{L^{3/2}} \leq C \sup_{1 \leq t < \infty} \|\rho\|_{L^\gamma}^{2\gamma/3} \leq C. \quad (5.11)$$

If  $\gamma > 3/2$ , since  $\rho_0 \in L^1$ , (1.1)<sub>1</sub> yields that for  $t \geq 0$ ,

$$\int \rho(x, t) dx = \int \rho_0(x) dx,$$

which combined with (5.10) implies

$$\sup_{1 \leq t < \infty} \|\rho\|_{L^{3/2}} \leq C \sup_{1 \leq t < \infty} \|\rho\|_{L^1}^{2/3} \leq C. \quad (5.12)$$

Similar to (3.40), one deduces from (1.1)<sub>2</sub> that

$$P = (-\Delta)^{-1} \operatorname{div}(\rho\dot{u}) + (2\mu + \lambda) \operatorname{div} u + (-\Delta)^{-1} \operatorname{div} \operatorname{div} \left( (H \otimes H) - \frac{1}{2} |H|^2 \right),$$

which together with the Sobolev inequality gives

$$\begin{aligned}
\|P\|_{L^2} &\leq C\|(-\Delta)^{-1}\operatorname{div}(\rho\dot{u})\|_{L^2} + C\|\nabla u\|_{L^2} + C\|H\|_{L^4}^2 \\
&\leq C\|\rho\dot{u}\|_{L^{6/5}} + C\|\nabla u\|_{L^2} + C\|H\|_{L^3}\|H\|_{L^6} \\
&\leq C\|\rho\|_{L^{3/2}}^{1/2}\|\rho^{1/2}\dot{u}\|_{L^2} + C\|\nabla u\|_{L^2} + C\|\nabla H\|_{L^2} \\
&\leq C\|\rho^{1/2}\dot{u}\|_{L^2} + C\|\nabla u\|_{L^2} + C\|\nabla H\|_{L^2},
\end{aligned}$$

owing to (5.10), (5.11) and (5.12). This combined with (5.10) leads to

$$\int_1^\infty \|P\|_{L^2}^2 dt \leq C. \quad (5.13)$$

For  $p \geq 2$ , we have similarly to (3.47) that

$$(\|P\|_{L^p}^p)_t + \frac{p\gamma - 1}{2\mu + \lambda} \|P\|_{L^{p+1}}^{p+1} = -\frac{p\gamma - 1}{2\mu + \lambda} \int P^p \left( F + \frac{1}{2}|H|^2 \right) dx, \quad (5.14)$$

which together with Holder's inequality yields

$$(\|P\|_{L^p}^p)_t + \frac{p\gamma - 1}{2(2\mu + \lambda)} \|P\|_{L^{p+1}}^{p+1} \leq C(p)\|F\|_{L^{p+1}}^{p+1} + C(p)\||H|^2\|_{L^{p+1}}^{p+1}. \quad (5.15)$$

Next, it follows from (1.1)<sub>3</sub> that

$$\frac{d}{dt} \|\nabla H\|_{L^2}^2 + (\|H_t\|_{L^2}^2 + \|\nabla^2 H\|_{L^2}^2) \leq C\|\nabla u\|_{L^2}^4 \|\nabla H\|_{L^2}^2, \quad (5.16)$$

and multiplying (1.1)<sub>3</sub> by  $H|H|^2$ , then integrating by parts over  $\mathbb{R}^3$  yields

$$\begin{aligned}
\frac{d}{dt} \|H\|_{L^4}^4 + \int (|H|^2 |\nabla H|^2 + |\nabla |H|^2|^2) dx &\leq C \int |\nabla u| |H|^4 dx \\
&\leq C\|\nabla u\|_{L^2} \||H|^2\|_{L^4}^2 \leq C\|\nabla u\|_{L^2} \||H|^2\|_{L^2}^{1/2} \||H|^2\|_{L^6}^{3/2} \\
&\leq C\|\nabla u\|_{L^2} \||H|^2\|_{L^2}^{1/2} \|\nabla |H|^2\|_{L^2}^{3/2} \leq \frac{1}{2} \|\nabla |H|^2\|_{L^2}^2 + C\|\nabla u\|_{L^2}^4 \|H\|_{L^4}^4
\end{aligned} \quad (5.17)$$

which together with (5.10), Gronwall's inequality yields that

$$\sup_{1 \leq t < \infty} \||H|^2\|_{L^2}^2 + \int_1^\infty \||H|\nabla H\|_{L^2}^2 dt \leq C. \quad (5.18)$$

Now, for  $B(t)$  defined as in (3.22), we have by (3.18), (5.16) and (5.7) that

$$\begin{aligned}
B'(t) &+ \|\rho^{1/2}\dot{u}\|_{L^2}^2 + \|H_t\|_{L^2}^2 + \|\nabla^2 H\|_{L^2}^2 \\
&\leq C\|P\|_{L^3}^3 + C\|\nabla u\|_{L^3}^3 + C\|H\|_{L^6}^6 + C\|\nabla u\|_{L^2}^4 \|\nabla H\|_{L^2}^2 \\
&\leq C\|P\|_{L^3}^3 + C\|H\|_{L^6}^6 + C\|\nabla u\|_{L^2}^4 \|\nabla H\|_{L^2}^2 \\
&\quad + (C\|F\|_{L^3}^3 + C\|\omega\|_{L^3}^3 + C\|P\|_{L^3}^3 + C\||H|^2\|_{L^3}^3) \\
&\leq C_1\|P\|_{L^3}^3 + C\|F\|_{L^3}^3 + C\|\omega\|_{L^3}^3 + C\|\nabla H\|_{L^2}^6 + C\|\nabla u\|_{L^2}^4 \|\nabla H\|_{L^2}^2.
\end{aligned} \quad (5.19)$$

Notice that  $B(t)$  satisfies (3.23), choosing  $C_2 \geq 2 + 2(2\mu + \lambda)(C_1 + 1)/(2\gamma - 1)$  and  $C_3$  suitably large such that

$$\begin{aligned}
\frac{\mu}{4} \|\nabla u\|_{L^2}^2 + \frac{\nu}{2} \|\nabla H\|_{L^2}^2 + \|H\|_{L^4}^4 + \|P\|_{L^2}^2 &\leq B(t) + C_2 \|P\|_{L^2}^2 + C_3 \|H\|_{L^4}^4 \\
&\leq C \|\nabla u\|_{L^2}^2 + C \|P\|_{L^2}^2 + C \|\nabla H\|_{L^2}^2 + C \|H\|_{L^4}^4.
\end{aligned} \quad (5.20)$$



Setting  $p = 2$  in (5.15), adding (5.15) multiplied by  $C_2$  and (5.17) multiplied by  $C_3$  to (5.19), yield that for  $t \geq 1$ ,

$$\begin{aligned}
& (B(t) + C_2\|P\|_{L^2}^2 + C_3\|H\|_{L^4}^4)' \\
& \quad + \|\rho^{\frac{1}{2}}\dot{u}\|_{L^2}^2 + \|H_t\|_{L^2}^2 + \|\nabla^2 H\|_{L^2}^2 + \|P\|_{L^3}^3 + \|H\|\|\nabla H\|_{L^2}^2 \\
& \leq C\|F\|_{L^3}^3 + C\|\omega\|_{L^3}^3 + C\|\nabla H\|_{L^2}^6 + C\|\nabla u\|_{L^2}^4\|\nabla H\|_{L^2}^2 + C\|\nabla u\|_{L^2}^4\|H\|_{L^4}^4 \\
& \leq C\left(\|\rho^{\frac{1}{2}}\dot{u}\|_{L^2} + \|H\|\|\nabla H\|_{L^2}\right)^{\frac{3}{2}}\left(\|\nabla u\|_{L^2} + \|P\|_{L^2} + \|H\|_{L^4}^2\right)^{\frac{3}{2}} \\
& \quad + C\|\nabla H\|_{L^2}^6 + C\|\nabla u\|_{L^2}^4\|\nabla H\|_{L^2}^2 + C\|\nabla u\|_{L^2}^4\|H\|_{L^4}^4 \\
& \leq \varepsilon\|\rho^{\frac{1}{2}}\dot{u}\|_{L^2}^2 + \varepsilon\|H\|\|\nabla H\|_{L^2}^2 + C\|\nabla u\|_{L^2}^6 + C\|P\|_{L^2}^6 + C\|H\|_{L^4}^{12} \\
& \quad + C\|\nabla H\|_{L^2}^6 + C\|\nabla u\|_{L^2}^4\|\nabla H\|_{L^2}^2 + C\|\nabla u\|_{L^2}^4\|H\|_{L^4}^4 \\
& \leq C\|\nabla u\|_{L^2}^4 + C\|\nabla H\|_{L^2}^4 + C\|P\|_{L^2}^4 + C\|H\|_{L^4}^8
\end{aligned} \tag{5.21}$$

where in the second inequality one has used (5.6), and in the last inequality one has used (5.10), (5.18) and chosen  $\varepsilon$  suitably small.

Multiplying (5.21) by  $t$ , along with Gronwall's inequality, (5.20), (5.10) and (5.13), gives

$$\begin{aligned}
& \sup_{1 \leq t < \infty} t\left(\|\nabla u\|_{L^2}^2 + \|\nabla H\|_{L^2}^2 + \|P\|_{L^2}^2 + \|H\|_{L^4}^4\right) \\
& \quad + \int_1^\infty t\left(\|\rho^{\frac{1}{2}}\dot{u}\|_{L^2}^2 + \|H_t\|_{L^2}^2 + \|\nabla^2 H\|_{L^2}^2 + \|P\|_{L^3}^3 + \|H\|\|\nabla H\|_{L^2}^2\right) dt \leq C.
\end{aligned} \tag{5.22}$$

Following the same arguments as (3.24), we deduce that

$$\begin{aligned}
& \frac{d}{dt} \int \rho|\dot{u}|^2 dx + \int |\nabla \dot{u}|^2 dx \leq \varepsilon\|\nabla \dot{u}\|_{L^2}^2 + \varepsilon\|\nabla H_t\|_{L^2}^2 + C\|\nabla u\|_{L^4}^4 + C\|P\|_{L^4}^4 \\
& \quad + C\|\nabla H\|_{L^2}^4\|H_t\|_{L^2}^2 + C\|\nabla u\|_{L^2}^2\|\nabla H\|_{L^2}^2\|\nabla^2 H\|_{L^2}^2.
\end{aligned} \tag{5.23}$$

Next, noting that

$$H_{tt} - \nu \Delta H_t = (H \cdot \nabla u - u \cdot \nabla H - H \operatorname{div} u)_t,$$

and using the fact that  $u_t = \dot{u} - u \cdot \nabla u$ , we obtain after direct computations that

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \int |H_t|^2 dx + \nu \int |\nabla H_t|^2 dx = \int (H_t \cdot \nabla u - u \cdot \nabla H_t - H_t \operatorname{div} u) H_t dx \\
& \quad + \int (H \cdot \nabla \dot{u} - \dot{u} \cdot \nabla H - H \operatorname{div} \dot{u}) H_t dx \\
& \quad - \int [H \cdot \nabla (u \cdot \nabla u) - (u \cdot \nabla u) \cdot \nabla H - H \operatorname{div} (u \cdot \nabla u)] H_t dx \\
& = \bar{I}_1 + \bar{I}_2 + \bar{I}_3,
\end{aligned} \tag{5.24}$$

where

$$\begin{aligned}
\bar{I}_1 & \leq C \int |H_t|^2 |\nabla u| dx \leq C\|H_t\|_{L^4}^2 \|\nabla u\|_{L^2} \\
& \leq C\|H_t\|_{L^2}^{\frac{1}{2}} \|\nabla H_t\|_{L^2}^{\frac{3}{2}} \|\nabla u\|_{L^2} \leq \varepsilon\|\nabla H_t\|_{L^2}^2 + C\|H_t\|_{L^2}^2 \|\nabla u\|_{L^2}^4 \\
\bar{I}_2 & \leq C\|\nabla \dot{u}\|_{L^2} \|\nabla H\|_{L^2} \|H_t\|_{L^3} \leq C\|\nabla \dot{u}\|_{L^2} \|\nabla H\|_{L^2} \|H_t\|_{L^2}^{\frac{1}{2}} \|\nabla H_t\|_{L^2}^{\frac{1}{2}} \\
& \leq \varepsilon\|\nabla \dot{u}\|_{L^2}^2 + \varepsilon\|\nabla H_t\|_{L^2}^2 + C\|H_t\|_{L^2}^2 \|\nabla H\|_{L^2}^4 \\
\bar{I}_3 & \leq C\|H\|_{L^{12}} \|u\|_{L^6} \|\nabla u\|_{L^4} \|\nabla H_t\|_{L^2} \leq C\|\nabla |H|^2\|_{L^2}^{1/2} \|\nabla u\|_{L^2} \|\nabla u\|_{L^4} \|\nabla H_t\|_{L^2} \\
& \leq \varepsilon\|\nabla H_t\|_{L^2}^2 + C\|H\|\|\nabla H\|_{L^2}^2 \|\nabla u\|_{L^2}^4 + C\|\nabla u\|_{L^4}^4.
\end{aligned} \tag{5.25}$$

Putting (5.25) into (5.24), adding the resulting inequality to (5.23), and choosing  $\varepsilon$  suitably small, gives

$$\begin{aligned} & \frac{d}{dt} \left( \int \rho |\dot{u}|^2 dx + \int |H_t|^2 dx \right) + \int |\nabla \dot{u}|^2 dx + \int |\nabla H_t|^2 dx \\ & \leq C \|\nabla u\|_{L^4}^4 + C \|P\|_{L^4}^4 \\ & \quad + C (\|\nabla H\|_{L^2}^4 + \|\nabla u\|_{L^2}^4) (\|H_t\|_{L^2}^2 + \|\nabla^2 H\|_{L^2}^2 + \|\nabla |H|^2\|_{L^2}^2) \end{aligned} \quad (5.26)$$

Notice that, by (5.8), (5.9) and (5.10), we have following estimates:

$$\|H\|_{L^4}^4 \leq C \|H\|_{L^3}^2 \|H\|_{L^6}^2 \leq C \|\nabla H\|_{L^2}^2 \quad (5.27)$$

$$\|H \cdot \nabla H\|_{L^2}^2 \leq C \|H\|_{L^2} \|\nabla H\|_{L^2}^2 \leq C \|H\|_{L^3}^2 \|\nabla H\|_{L^6}^2 \leq C \|\nabla^2 H\|_{L^2}^2 \quad (5.28)$$

$$\begin{aligned} \|\nabla^2 H\|_{L^2}^2 & \leq C \|H_t\|_{L^2}^2 + C \|u \cdot \nabla H\|_{L^2}^2 + C \|H\|_{L^2} \|\nabla u\|_{L^2}^2 \\ & \leq C \|H_t\|_{L^2}^2 + C \|u\|_{L^6}^2 \|\nabla H\|_{L^3}^2 + C \|H\|_{L^\infty}^2 \|\nabla u\|_{L^2}^2 \\ & \leq C \|H_t\|_{L^2}^2 + C \|\nabla u\|_{L^2}^2 \|\nabla H\|_{L^2} \|\nabla^2 H\|_{L^2} \\ & \leq C \|H_t\|_{L^2}^2 + C \|\nabla u\|_{L^2}^4 \|\nabla H\|_{L^2}^2 + \frac{1}{2} \|\nabla^2 H\|_{L^2}^2 \\ & \leq C \|H_t\|_{L^2}^2 + C \|\nabla u\|_{L^2}^4 + C \|\nabla H\|_{L^2}^4 + \frac{1}{2} \|\nabla^2 H\|_{L^2}^2. \end{aligned} \quad (5.29)$$

Now, by (5.7), (5.6), (5.29), (5.28) and (5.27), it holds that

$$\begin{aligned} \|\nabla u\|_{L^4}^4 & \leq C \|F\|_{L^4}^4 + C \|\omega\|_{L^4}^4 + C \|P\|_{L^4}^4 + C \|H\|_{L^4}^4 \\ & \leq C \|P\|_{L^4}^4 + C (\|\rho \dot{u}\|_{L^2} + \|H\|_{L^2} \|\nabla H\|_{L^2})^3 (\|\nabla u\|_{L^2} + \|P\|_{L^2} + \|H\|_{L^4}^2) \\ & \leq C \|P\|_{L^4}^4 + C \left( \|\rho^{\frac{1}{2}} \dot{u}\|_{L^2}^2 + \|H\|_{L^2} \|\nabla H\|_{L^2}^2 \right) \\ & \quad \cdot \left( \|\rho^{\frac{1}{2}} \dot{u}\|_{L^2}^2 + \|H\|_{L^2} \|\nabla H\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \|P\|_{L^2}^2 + \|H\|_{L^4}^4 \right) \\ & \leq C \|P\|_{L^4}^4 + C \left( \|\rho^{\frac{1}{2}} \dot{u}\|_{L^2}^2 + \|H_t\|_{L^2}^2 \right) \Phi(t) + C (\|\nabla H\|_{L^2}^4 + \|\nabla u\|_{L^2}^4) \Phi(t) \end{aligned} \quad (5.30)$$

where

$$\Phi(t) \triangleq \|\rho^{\frac{1}{2}} \dot{u}\|_{L^2}^2 + \|H\|_{L^2} \|\nabla H\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \|P\|_{L^2}^2 + \|\nabla H\|_{L^2}^2. \quad (5.31)$$

Submitting (5.30) into (5.26), we have

$$\begin{aligned} & \frac{d}{dt} \left( \int \rho |\dot{u}|^2 dx + \int |H_t|^2 dx \right) + \int |\nabla \dot{u}|^2 dx + \int |\nabla H_t|^2 dx \\ & \leq \tilde{C} \|P\|_{L^4}^4 + C \left( \|\rho^{\frac{1}{2}} \dot{u}\|_{L^2}^2 + \|H_t\|_{L^2}^2 \right) \Phi(t) \\ & \quad + C (\|\nabla H\|_{L^2}^4 + \|\nabla u\|_{L^2}^4) (\Phi(t) + \|H_t\|_{L^2}^2 + \|\nabla^2 H\|_{L^2}^2) \end{aligned} \quad (5.32)$$

Setting  $p = 3$  in (5.15), and adding (5.15) multiplied by  $2(2\mu + \lambda)(\tilde{C} + 1)/(3\gamma - 1)$  to (5.32), then multiplying the resulting inequality by  $t^2$ , lead to

$$\begin{aligned} & \frac{d}{dt} \left( t^2 \|\rho^{\frac{1}{2}} \dot{u}\|_{L^2}^2 + t^2 \|H_t\|_{L^2}^2 + \frac{2(2\mu + \lambda)(\tilde{C} + 1)}{3\gamma - 1} t^2 \|P\|_{L^3}^3 \right) \\ & \quad + t^2 \|\nabla \dot{u}\|_{L^2}^2 + t^2 \|\nabla H_t\|_{L^2}^2 + t^2 \|P\|_{L^4}^4 \\ & \leq C \left( t^2 \|\rho^{\frac{1}{2}} \dot{u}\|_{L^2}^2 + t^2 \|H_t\|_{L^2}^2 \right) \Phi(t) \\ & \quad + Ct \left( \|\rho^{\frac{1}{2}} \dot{u}\|_{L^2}^2 + \|H_t\|_{L^2}^2 + \|P\|_{L^3}^3 \right) + C (\Phi(t) + \|H_t\|_{L^2}^2 + \|\nabla^2 H\|_{L^2}^2) \end{aligned} \quad (5.33)$$

where in the last inequality one has used (5.22). This combined with Gronwall's inequality, (5.22), (5.10), (5.13) and (5.18), yields that

$$\sup_{1 \leq t < \infty} t^2 \int (\rho |\dot{u}|^2 + |H_t|^2 + P^3) dx + \int_1^\infty t^2 (\|\nabla \dot{u}\|_{L^2}^2 + \|P\|_{L^4}^4 + \|\nabla H_t\|_{L^2}^2) dt \leq C. \quad (5.34)$$

Moreover, we have by (5.29), (5.34) and (5.22) that

$$\sup_{1 \leq t < \infty} t^2 \|\nabla^2 H\|_{L^2}^2 \leq C \sup_{1 \leq t < \infty} t^2 \|H_t\|_{L^2}^2 + \sup_{1 \leq t < \infty} t^2 (\|\nabla H\|_{L^2}^4 + \|\nabla u\|_{L^2}^4) \leq C, \quad (5.35)$$

which combined with (5.28) gives

$$\sup_{1 \leq t < \infty} t^2 \|H\| \|\nabla H\|_{L^2}^2 \leq C \sup_{1 \leq t < \infty} t^2 \|\nabla^2 H\|_{L^2}^2 \leq C. \quad (5.36)$$

Then, (5.22), (5.34), (5.35) and (5.36) combined with (5.7) gives (1.24) provided we show that for  $m = 1, 2, \dots$ ,

$$\sup_{1 \leq t < \infty} t^m \|P\|_{L^{m+1}}^{m+1} + \int_0^\infty t^m \|P\|_{L^{m+2}}^{m+2} dt \leq C(m). \quad (5.37)$$

Finally, we need only to prove (5.37). Since (5.22) shows that (5.37) holds for  $m = 1$ , we assume that (5.37) holds for  $m = n$ , that is,

$$\sup_{1 \leq t < \infty} t^n \|P\|_{L^{n+1}}^{n+1} + \int_1^\infty t^n \|P\|_{L^{n+2}}^{n+2} dt \leq C(n). \quad (5.38)$$

Setting  $p = n + 2$  in (5.14) and multiplying (5.14) by  $t^{n+1}$  give

$$\begin{aligned} & \frac{2(2\mu + \lambda)}{(n+2)\gamma - 1} (t^{n+1} \|P\|_{L^{n+2}}^{n+2})_t + t^{n+1} \|P\|_{L^{n+3}}^{n+3} \\ & \leq C(n) t^n \|P\|_{L^{n+2}}^{n+2} + C(n) t^{n+1} \|P\|_{L^{n+2}}^{n+2} (\|F\|_{L^\infty} + \| |H|^2 \|_{L^\infty}). \end{aligned} \quad (5.39)$$

It follows from (5.8)-(5.9), (5.5), (5.22), (5.34), (5.35) and (5.36) that

$$\begin{aligned} & \int_1^\infty (\|F\|_{L^\infty} + \| |H|^2 \|_{L^\infty}) dt \\ & \leq C \int_1^\infty \|F\|_{L^6}^{1/2} \|\nabla F\|_{L^6}^{1/2} dt + C \int_1^\infty \| |H|^2 \|_{L^6}^{1/2} \|\nabla |H|^2 \|_{L^6}^{1/2} dt \\ & \leq C \int_1^\infty \|\nabla F\|_{L^2}^{1/2} \|\nabla F\|_{L^6}^{1/2} dt + C \int_1^\infty \| |H| \|\nabla H\|_{L^2}^{1/2} \| |H| \|\nabla H\|_{L^6}^{1/2} dt \\ & \leq C \int_1^\infty \left( \|\rho^{1/2} \dot{u}\|_{L^2} + \| |H| \|\nabla H\|_{L^2} \right)^{1/2} \left( \|\rho^{1/2} \dot{u}\|_{L^6} + \| |H| \|\nabla H\|_{L^6} \right)^{1/2} dt \\ & \leq C \int_1^\infty t^{-1/2} (\|\dot{u}\|_{L^6} + \|H\|_{L^\infty} \|\nabla H\|_{L^6})^{1/2} dt \\ & \leq C \int_1^\infty t^{-1/2} \left( \|\nabla \dot{u}\|_{L^2} + \|H\|_{L^6}^{1/2} \|\nabla H\|_{L^6}^{1/2} \|\nabla^2 H\|_{L^2} \right)^{1/2} dt \\ & \leq C \int_1^\infty t^{-4/3} dt + C \int_1^\infty t^2 \|\nabla \dot{u}\|_{L^2}^2 dt + C \int_1^\infty t^{-1/2} t^{-1/8} t^{-3/4} dt \\ & \leq C \end{aligned}$$

which, along with (5.39), (5.38), and Gronwall's inequality, thus shows that (5.37) holds for  $m = n + 1$ . By induction, we obtain (5.37) and finish the proof of (1.24). The proof of Theorem 1.2 is completed.

## References

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